The Online Appendix for Benefits of FDI subsidies: The role of funding sources

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This technical appendix provides total equilibrium conditions and mathematical derivations. We only show the characterization of our model with the separable utility since the equilibrium under the CES utility can be obtained analogously. Note that all tax variables in the online appendix are expressed as wedge terms for algebraic simplicity. The following table displays the mapping of tax variables from the main text to the online appendix.

Table 1: Notations for tax variables in the online appendix

Main Text	Online Appendix
Labor-income tax rate τ_L	Labor-income tax wedge $\tau'_L = 1 - \tau_L$
Consumption tax rate τ_C	Consumption tax wedge $\tau'_C = 1 + \tau_C$
Firm-revenue tax rate τ_R	Firm-revenue tax wedge $\tau'_R = 1 - \tau_R$

In what follows, we use τ'_L, τ'_C, and τ'_R and suppress the superscript '$\prime$' in all equilibrium conditions. By contrast, the notation for a FDI subsidy, s_V, indicates a rate as in the main text.

1 Total Equilibrium Conditions

We have 15 equations to solve for 15 variables: C^H, C^{H*}, C^{F*}, C^F, V, V^*, Z^D, Z^X, Z^I, Z^{D*}, Z^{X*}, Z^{I*}, M, M^*, and one tax variable from $\{\tau_L, \tau_C, \tau_R\}$. There is no government in Foreign: $s_V^* = 0$ and $\tau_L^* = \tau_C^* = \tau_R^* = 1$. We take the partial equilibrium analysis by excluding labor market clearing conditions: $\frac{W^*}{W} = 1$. $\frac{W}{P_0} = \frac{W^*}{P_0^*} = 1$ holds due to the CRTS technology of homogeneous goods. Home homogeneous good is the numeraire in Home and Foreign homogeneous good is the numeraire in Foreign: $P_0 = P_0^* = 1$.

Market Demand Shifters:

$$A^{H} \equiv (\tau_C)^{-\sigma}\left(C^{H}\right)^{\theta\sigma+1-\sigma}, \qquad A^{H*} \equiv (\tau_C^*)^{-\sigma}\left(C^{H*}\right)^{\theta\sigma+1-\sigma},$$

$$A^{F*} \equiv (\tau_C^*)^{-\sigma}\left(C^{F*}\right)^{\theta\sigma+1-\sigma}, \quad A^{F} \equiv (\tau_C)^{-\sigma}\left(C^{F}\right)^{\theta\sigma+1-\sigma}.$$

Average Productivity and Useful Definitions under the Pareto distribution:

$$\frac{J(Z^D,Z^X)}{G(Z^X)-G(Z^D)} = \left(\widetilde{Z}^L\right)^{\sigma-1} = \left(\frac{\eta}{\eta-\sigma+1}\right)\left(\frac{\left(Z^D\right)^{-(\eta-\sigma+1)}-\left(Z^X\right)^{-(\eta-\sigma+1)}}{(Z^D)^{-\eta}-(Z^X)^{-\eta}}\right),$$

$$\frac{J(Z^X,Z^I)}{G(Z^I)-G(Z^X)} = \left(\widetilde{Z}^X\right)^{\sigma-1} = \left(\frac{\eta}{\eta-\sigma+1}\right)\left(\frac{\left(Z^X\right)^{-(\eta-\sigma+1)}-\left(Z^I\right)^{-(\eta-\sigma+1)}}{(Z^X)^{-\eta}-(Z^I)^{-\eta}}\right),$$

$$\frac{J(Z^I)}{1-G(Z^I)} = \left(\widetilde{Z}^I\right)^{\sigma-1} = \left(\frac{\eta}{\eta-\sigma+1}\right)\left(Z^I\right)^{\sigma-1},$$

$$\frac{J(Z^D)}{1-G(Z^D)} = \left(\widetilde{Z}^D\right)^{\sigma-1} = \left(\frac{\eta}{\eta-\sigma+1}\right)\left(Z^D\right)^{\sigma-1}.$$

$$\frac{J^*(Z^{D*},Z^{X*})}{G^*(Z^{X*})-G^*(Z^{D*})} = \left(\widetilde{Z}^{L*}\right)^{\sigma-1} = \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{\left(Z^{D*}\right)^{-(\eta^*-\sigma+1)}-\left(Z^{X*}\right)^{-(\eta^*-\sigma+1)}}{(Z^{D*})^{-\eta^*}-(Z^{X*})^{-\eta^*}}\right),$$

$$\frac{J^*(Z^{X*},Z^{I*})}{G^*(Z^{I*})-G^*(Z^{X*})} = \left(\widetilde{Z}^{X*}\right)^{\sigma-1} = \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{\left(Z^{X*}\right)^{-(\eta^*-\sigma+1)}-\left(Z^{I*}\right)^{-(\eta^*-\sigma+1)}}{(Z^{X*})^{-\eta^*}-(Z^{I*})^{-\eta^*}}\right),$$

$$\frac{J^*(Z^{I*})}{1-G^*(Z^{I*})} = \left(\widetilde{Z}^{I*}\right)^{\sigma-1} = \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(Z^{I*}\right)^{\sigma-1},$$

$$\frac{J^*(Z^{D*})}{1-G^*(Z^{D*})} = \left(\widetilde{Z}^{D*}\right)^{\sigma-1} = \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(Z^{D*}\right)^{\sigma-1}.$$

These are equivalent to

$$\frac{J(Z^D,Z^X)}{1-G(Z^D)} = \int_{Z^D}^{Z^X} z^{\sigma-1}\frac{dG(z)}{1-G(Z^D)} = \left(\frac{\eta}{\eta-\sigma+1}\right)\left(\frac{\left(Z^D\right)^{-\eta+\sigma-1}-\left(Z^X\right)^{-\eta+\sigma-1}}{(Z^D)^{-\eta}}\right),$$

$$\frac{J(Z^X,Z^I)}{1-G(Z^D)} = \int_{Z^X}^{Z^I} z^{\sigma-1}\frac{dG(z)}{1-G(Z^D)} = \left(\frac{\eta}{\eta-\sigma+1}\right)\left(\frac{\left(Z^X\right)^{-\eta+\sigma-1}-\left(Z^I\right)^{-\eta+\sigma-1}}{(Z^D)^{-\eta}}\right),$$

$$\frac{J(Z^I)}{1-G(Z^D)} = \int_{Z^I}^{\infty} z^{\sigma-1}\frac{dG(z)}{1-G(Z^D)} = \left(\frac{\eta}{\eta-\sigma+1}\right)\left(\frac{\left(Z^I\right)^{-\eta+\sigma-1}}{(Z^D)^{-\eta}}\right),$$

$$\frac{J(Z^D)}{1-G(Z^D)} = \int_{Z^D}^{\infty} z^{\sigma-1}\frac{dG(z)}{1-G(Z^D)} = \left(\frac{\eta}{\eta-\sigma+1}\right)\left(Z^D\right)^{\sigma-1}.$$

$$\frac{J^*(Z^{D*},Z^{X*})}{1-G^*(Z^{D*})} = \int_{Z^{D*}}^{Z^{X*}} z^{\sigma-1}\frac{dG^*(z)}{1-G^*(Z^{D*})} = \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{\left(Z^{D*}\right)^{-\eta^*+\sigma-1}-\left(Z^{X*}\right)^{-\eta^*+\sigma-1}}{(Z^{D*})^{-\eta^*}}\right),$$

$$\frac{J^*(Z^{X*},Z^{I*})}{1-G^*(Z^{D*})} = \int_{Z^{X*}}^{Z^{I*}} z^{\sigma-1}\frac{dG^*(z)}{1-G^*(Z^{D*})} = \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{\left(Z^{X*}\right)^{-\eta^*+\sigma-1}-\left(Z^{I*}\right)^{-\eta^*+\sigma-1}}{(Z^{D*})^{-\eta^*}}\right),$$

$$\frac{J^*(Z^{I*})}{1-G^*(Z^{D*})} = \int_{Z^{I*}}^{\infty} z^{\sigma-1}\frac{dG^*(z)}{1-G^*(Z^{D*})} = \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{\left(Z^{I*}\right)^{-\eta^*+\sigma-1}}{(Z^{D*})^{-\eta^*}}\right),$$

$$\frac{J^*(Z^{D*})}{1-G^*(Z^{D*})} = \int_{Z^{D*}}^{\infty} z^{\sigma-1}\frac{dG^*(z)}{1-G^*(Z^{D*})} = \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(Z^{D*}\right)^{\sigma-1}.$$

Home Households (Equation 1, 2):

$$
\begin{aligned}
&\left(C^H\right)^{(\sigma-1)(1-\theta)} \\
&= M^L \int_{Z^D}^{Z^X} \left[\frac{\tau_C p^D(z)}{P_0}\right]^{1-\sigma} \frac{dG(z)}{G(Z^X)-G(Z^D)} + M^X \int_{Z^X}^{Z^I} \left[\frac{\tau_C p^{D,X}(z)}{P_0}\right]^{1-\sigma} \frac{dG(z)}{G(Z^I)-G(Z^X)} + M^I \int_{Z^I}^{\infty} \left[\frac{\tau_C p^{D,I}(z)}{P_0}\right]^{1-\sigma} \frac{dG(z)}{1-G(Z^I)} \\
&= \frac{M}{1-G(Z^D)} \int_{Z^D}^{\infty} \left[(\tau_C)\left(\frac{1}{\rho}\right)\left(\frac{1}{\tau_R}\right)\left(\frac{W}{P_0}\right)\left(\frac{1}{z}\right)\right]^{1-\sigma} dG(z) \\
&= \frac{M}{1-G(Z^D)} (\tau_C)^{1-\sigma} \left(\frac{1}{\rho}\right)^{1-\sigma} \left(\frac{1}{\tau_R}\right)^{1-\sigma} \left(\frac{W}{P_0}\right)^{1-\sigma} J(Z^D) \\
&= \underbrace{M}_{\text{Variety}} \underbrace{\left(\frac{1}{\rho}\frac{W}{P_0}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}}_{\text{Effective Price}} \underbrace{\left(\widetilde{Z}^D\right)^{\sigma-1}}_{\text{Average Productivity}}
\end{aligned}
$$

where $\sigma > 1$, $M^L = \left(\frac{G(Z^X)-G(Z^D)}{1-G(Z^D)}\right) M$, $M^X = \left(\frac{G(Z^I)-G(Z^X)}{1-G(Z^D)}\right) M$, and $M^I = \left(\frac{1-G(Z^I)}{1-G(Z^D)}\right) M$.

$$
\begin{aligned}
&\left(C^F\right)^{(\sigma-1)(1-\theta)} \\
&= M^{X*} \int_{Z^{X*}}^{Z^{I*}} \left[\frac{\tau_C p^X(z)}{P_0}\right]^{1-\sigma} \frac{dG^*(z)}{G^*(Z^{I*})-G^*(Z^{X*})} + M^{I*} \int_{Z^{I*}}^{\infty} \left[\frac{\tau_C p^I(z)}{P_0}\right]^{1-\sigma} \frac{dG^*(z)}{1-G^*(Z^{I*})} \\
&= \left(\begin{array}{ll} + & M^{X*}\left[(\tau_C)\left(\frac{\tau^*}{\rho}\right)\left(\frac{1}{\tau_R^*}\right)\left(\frac{W^*}{P_0}\right)\right]^{1-\sigma} \frac{J^*\left(Z^{X*},Z^{I*}\right)}{G^*(Z^{I*})-G^*(Z^{X*})} \\ + & M^{I*}\left[(\tau_C)\left(\frac{1-s_V}{\rho}\right)\left(\frac{1}{\tau_R}\right)\left(\frac{W}{P_0}\right)\right]^{1-\sigma} \frac{J^*(Z^{I*})}{1-G^*(Z^{I*})} \end{array}\right) \\
&= \left(\begin{array}{ll} + & \frac{M^*}{1-G^*(Z^{D*})} (\tau_C)^{1-\sigma} \left(\frac{\tau^*}{\rho}\right)^{1-\sigma} \left(\frac{1}{\tau_R^*}\right)^{1-\sigma} \left(\frac{W^*}{P_0}\right)^{1-\sigma} J^*\left(Z^{X*}, Z^{I*}\right) \\ + & \frac{M^*}{1-G^*(Z^{D*})} (\tau_C)^{1-\sigma} \left(\frac{1-s_V}{\rho}\right)^{1-\sigma} \left(\frac{1}{\tau_R}\right)^{1-\sigma} \left(\frac{W}{P_0}\right)^{1-\sigma} J^*(Z^{I*}) \end{array}\right) \\
&= \left(\begin{array}{ll} + & M^{X*}\left(\frac{\tau^*}{\rho}\frac{W^*}{P_0}\frac{\tau_C}{\tau_R^*}\right)^{-(\sigma-1)} \left(\widetilde{Z}^{X*}\right)^{\sigma-1} \\ + & \underbrace{M^{I*}}_{\text{Variety}} \underbrace{\left(\frac{1-s_V}{\rho}\frac{W}{P_0}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}}_{\text{Effective Price}} \underbrace{\left(\widetilde{Z}^{I*}\right)^{\sigma-1}}_{\text{Average Productivity}} \end{array}\right)
\end{aligned}
$$

where $\sigma > 1$, $M^{L*} = \left(\frac{G^*(Z^{X*})-G^*(Z^{D*})}{1-G^*(Z^{D*})}\right) M^*$, $M^{X*} = \left(\frac{G^*(Z^{I*})-G^*(Z^{X*})}{1-G^*(Z^{D*})}\right) M^*$, and $M^{I*} = \left(\frac{1-G^*(Z^{I*})}{1-G^*(Z^{D*})}\right) M^*$.

Foreign Households (Equation 3, 4):

$$
\begin{aligned}
&\left(C^{F*}\right)^{(\sigma-1)(1-\theta)} \\
&= \left(\begin{array}{l} \quad M^{L*}\int_{Z^{D*}}^{Z^{X*}}\left[\frac{\tau_C^* p^{D*}(z)}{P_0^*}\right]^{1-\sigma}\frac{dG^*(z)}{G^*(Z^{X*})-G^*(Z^{D*})} \\ + \quad M^{X*}\int_{Z^{X*}}^{Z^{I*}}\left[\frac{\tau_C^* p^{D,X*}(z)}{P_0^*}\right]^{1-\sigma}\frac{dG^*(z)}{G^*(Z^{I*})-G^*(Z^{X*})} + M^{I*}\int_{Z^{I*}}^{\infty}\left[\frac{\tau_C^* p^{D,I*}(z)}{P_0^*}\right]^{1-\sigma}\frac{dG^*(z)}{1-G^*(Z^{I*})} \end{array}\right) \\
&= \frac{M^*}{1-G^*(Z^{D*})}\int_{Z^{D*}}^{\infty}\left[\left(\tau_C^*\right)\left(\frac{1}{\rho}\right)\left(\frac{1}{\tau_R^*}\right)\left(\frac{W^*}{P_0^*}\right)\left(\frac{1}{z}\right)\right]^{1-\sigma}dG^*(z) \\
&= \frac{M^*}{1-G^*(Z^{D*})}\left(\tau_C^*\right)^{1-\sigma}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R^*}\right)^{1-\sigma}\left(\frac{W^*}{P_0^*}\right)^{1-\sigma}J^*(Z^{D*}) \\
&= \underbrace{M^*}_{\text{Variety}}\underbrace{\left(\frac{1}{\rho}\frac{W^*}{P_0^*}\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)}}_{\text{Effective Price}}\underbrace{\left(\widetilde{Z}^{D*}\right)^{\sigma-1}}_{\text{Average Productivity}}
\end{aligned}
$$

where $\sigma > 1$, $M^{L*} = \left(\frac{G^*(Z^{X*})-G^*(Z^{D*})}{1-G^*(Z^{D*})}\right)M^*$, $M^{X*} = \left(\frac{G^*(Z^{I*})-G^*(Z^{X*})}{1-G^*(Z^{D*})}\right)M^*$, and $M^{I*} = \left(\frac{1-G^*(Z^{I*})}{1-G^*(Z^{D*})}\right)M^*$.

$$
\begin{aligned}
&\left(C^{H*}\right)^{-(1-\sigma)(1-\theta)} \\
&= M^X\int_{Z^X}^{Z^I}\left[\frac{\tau_C^* p^{X*}(z)}{P_0^*}\right]^{1-\sigma}\frac{dG(z)}{G(Z^I)-G(Z^X)} + M^I\int_{Z^I}^{\infty}\left[\frac{\tau_C^* p^{I*}(z)}{P_0^*}\right]^{1-\sigma}\frac{dG(z)}{1-G(Z^I)} \\
&= \left(\begin{array}{l} + \quad M^X\left[\left(\tau_C^*\right)\left(\frac{\tau}{\rho}\right)\left(\frac{1}{\tau_R}\right)\left(\frac{W}{P_0^*}\right)\right]^{1-\sigma}\frac{J\left(Z^X,Z^I\right)}{G(Z^I)-G(Z^X)} \\ + \quad M^I\left[\left(\tau_C^*\right)\left(\frac{(1-s_V^*)}{\rho}\right)\left(\frac{1}{\tau_R^*}\right)\left(\frac{W^*}{P_0^*}\right)\right]^{1-\sigma}\frac{J(Z^I)}{1-G(Z^I)} \end{array}\right) \\
&= \left(\begin{array}{l} + \quad \frac{M}{1-G(Z^D)}\left(\tau_C^*\right)^{1-\sigma}\left(\frac{\tau}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}\left(\frac{W}{P_0^*}\right)^{1-\sigma}J\left(Z^X,Z^I\right) \\ + \quad \frac{M}{1-G(Z^D)}\left(\tau_C^*\right)^{1-\sigma}\left(\frac{1-s_V^*}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R^*}\right)^{1-\sigma}\left(\frac{W^*}{P_0^*}\right)^{1-\sigma}J(Z^I) \end{array}\right) \\
&= \left(\begin{array}{l} + \quad M^X\left(\frac{\tau}{\rho}\frac{W}{P_0^*}\frac{\tau_C^*}{\tau_R}\right)^{-(\sigma-1)}\left(\widetilde{Z}^X\right)^{\sigma-1} \\ + \quad \underbrace{M^I}_{\text{Variety}}\underbrace{\left(\frac{1-s_V^*}{\rho}\frac{W^*}{P_0^*}\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)}}_{\text{Effective Price}}\underbrace{\left(\widetilde{Z}^I\right)^{\sigma-1}}_{\text{Average Productivity}} \end{array}\right)
\end{aligned}
$$

where $\sigma > 1$, $M^L = \left(\frac{G(Z^X)-G(Z^D)}{1-G(Z^D)}\right)M$, $M^X = \left(\frac{G(Z^I)-G(Z^X)}{1-G(Z^D)}\right)M$, and $M^I = \left(\frac{1-G(Z^I)}{1-G(Z^D)}\right)M$.

The Indirect Utility (Equation 5, 6):

$$
\begin{aligned}
V &= \frac{W}{P_0}L\tau_L + \left(\frac{1}{\theta}-1\right)\left(C^H\right)^{\theta} + \left(\frac{1}{\theta}-1\right)\left(C^F\right)^{\theta} \\
V^* &= \frac{W^*}{P_0^*}L^*\tau_L^* + \left(\frac{1}{\theta}-1\right)\left(C^{F*}\right)^{\theta} + \left(\frac{1}{\theta}-1\right)\left(C^{H*}\right)^{\theta}
\end{aligned}
$$

Cutoff Productivity (Equation 7, 8, 9, 10, 11, 12):

$$Z^D = \left(\frac{\frac{W}{P_0} f^D}{(\tau_R)^\sigma (\frac{1}{\sigma}) (\frac{1}{\rho})^{1-\sigma} \left(\frac{W}{P_0}\right)^{1-\sigma} A^H} \right)^{\frac{1}{\sigma-1}}$$

$$Z^X = \left(\frac{\frac{W}{P_0^*} f^X}{(\tau_R)^\sigma (\frac{1}{\tau})^{\sigma-1} (\frac{1}{\sigma}) (\frac{1}{\rho})^{1-\sigma} (\frac{W}{W^*})^{1-\sigma} \left(\frac{W^*}{P_0^*}\right)^{1-\sigma} A^{H*}} \right)^{\frac{1}{\sigma-1}}$$

$$Z^I = \left(\frac{\frac{W^*}{P_0^*} f^{I*} - \frac{W}{P_0^*} f^X}{(\frac{1}{\sigma}) (\frac{1}{\rho})^{1-\sigma} \left[(\tau_R^*)^\sigma (1-s_V^*)^{1-\sigma} \left(\frac{W^*}{P_0^*}\right)^{1-\sigma} A^{H*} - (\tau_R)^\sigma (\frac{1}{\tau})^{\sigma-1} (\frac{W}{W^*})^{1-\sigma} \left(\frac{W^*}{P_0^*}\right)^{1-\sigma} A^{H*} \right]} \right)^{\frac{1}{\sigma-1}}$$

$$Z^{D*} = \left(\frac{\left(\frac{W^*}{P_0^*}\right) f^{D*}}{(\tau_R^*)^\sigma (\frac{1}{\sigma}) (\frac{1}{\rho})^{1-\sigma} \left(\frac{W^*}{P_0^*}\right)^{1-\sigma} A^{F*}} \right)^{\frac{1}{\sigma-1}}$$

$$Z^{X*} = \left(\frac{\frac{W^*}{P_0} f^{X*}}{(\tau_R^*)^\sigma (\frac{1}{\tau^*})^{\sigma-1} (\frac{1}{\sigma}) (\frac{1}{\rho})^{1-\sigma} \left(\frac{W^*}{W}\right)^{1-\sigma} \left(\frac{W}{P_0}\right)^{1-\sigma} A^F} \right)^{\frac{1}{\sigma-1}}$$

$$Z^{I*} = \left(\frac{\frac{W}{P_0} f^I - \frac{W^*}{P_0} f^{X*}}{(\frac{1}{\sigma}) (\frac{1}{\rho})^{1-\sigma} \left[(\tau_R)^\sigma (1-s_V)^{1-\sigma} \left(\frac{W}{P_0}\right)^{1-\sigma} A^F - (\tau_R^*)^\sigma (\frac{1}{\tau^*})^{\sigma-1} \left(\frac{W^*}{W}\right)^{1-\sigma} \left(\frac{W}{P_0}\right)^{1-\sigma} A^F \right]} \right)^{\frac{1}{\sigma-1}}$$

Free Entries (Equation 13, 14):

$$\delta \frac{W}{P_0} F^D = \left(\begin{array}{ll} + & J\left(Z^D\right) (\tau_R)^\sigma \left(\frac{1}{\sigma}\right) \left(\frac{1}{\rho}\right)^{1-\sigma} \left[\left(\frac{W}{P_0}\right)^{1-\sigma} (\tau_C)^{-\sigma} \left(C^H\right)^{\theta\sigma+1-\sigma} \right] \\ + & \left(\frac{P_0^*}{P_0}\right) J\left(Z^X, Z^I\right) (\tau_R)^\sigma \left(\frac{1}{\sigma}\right) \left(\frac{1}{\rho}\right)^{1-\sigma} \left(\frac{1}{\tau}\right)^{\sigma-1} \left[\left(\frac{W}{P_0^*}\right)^{1-\sigma} (\tau_C^*)^{-\sigma} \left(C^{H*}\right)^{\theta\sigma+1-\sigma} \right] \\ + & \left(\frac{P_0^*}{P_0}\right) J\left(Z^I\right) (\tau_R^*)^\sigma \left(\frac{1}{\sigma}\right) \left(\frac{1}{\rho}\right)^{1-\sigma} (1-s_V^*)^{1-\sigma} \left[\left(\frac{W^*}{P_0^*}\right)^{1-\sigma} (\tau_C^*)^{-\sigma} \left(C^{H*}\right)^{\theta\sigma+1-\sigma} \right] \\ - & \frac{W}{P_0} f^D \left(1 - G(Z^D)\right) \\ - & \frac{W}{P_0} f^X \left(G(Z^I) - G(Z^X)\right) \\ - & \left(\frac{P_0^*}{P_0}\right) \frac{W^*}{P_0^*} f^{I*} \left(1 - G(Z^I)\right) \end{array} \right)$$

$$\delta\frac{W^*}{P_0^*}F^{D*}=\left(\begin{array}{ll} + & J^*\left(Z^{D*}\right)\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\sigma}\right)\left(\frac{1}{\rho}\right)^{1-\sigma}\left[\left(\frac{W^*}{P_0^*}\right)^{1-\sigma}\left(\tau_C^*\right)^{-\sigma}\left(C^{F*}\right)^{\theta\sigma+1-\sigma}\right] \\ + & \left(\frac{P_0}{P_0^*}\right)J^*\left(Z^{X*},Z^{I*}\right)\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\sigma}\right)\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{W^*}{W}\right)^{1-\sigma}\left[\left(\frac{W}{P_0}\right)^{1-\sigma}\left(\tau_C\right)^{-\sigma}\left(C^{F}\right)^{\theta\sigma+1-\sigma}\right] \\ + & \left(\frac{P_0}{P_0^*}\right)J^*\left(Z^{I*}\right)\left(\tau_R\right)^{\sigma}\left(\frac{1}{\sigma}\right)\left(\frac{1}{\rho}\right)^{1-\sigma}\left(1-s_V\right)^{1-\sigma}\left[\left(\frac{W}{P_0}\right)^{1-\sigma}\left(\tau_C\right)^{-\sigma}\left(C^{F}\right)^{\theta\sigma+1-\sigma}\right] \\ - & \frac{W^*}{P_0^*}f^{D*}\left(1-G^*(Z^{D*})\right) \\ - & \frac{W^*}{P_0^*}f^{X*}\left(G^*(Z^{I*})-G^*(Z^{X*})\right) \\ - & \left(\frac{P_0}{P_0^*}\right)\frac{W}{P_0}f^{I}\left(1-G^*(Z^{I*})\right) \end{array}\right)$$

Mass of Firms:

$$M^E=\frac{\delta M}{1-G(Z^D)} \quad , \quad M^{E*}=\frac{\delta M^*}{1-G^*(Z^{D*})}$$

$$M^L=\left(\frac{G(Z^X)-G(Z^D)}{1-G(Z^D)}\right)M=\left(\frac{(Z^D)^{-\eta}-(Z^X)^{-\eta}}{(Z^D)^{-\eta}}\right)M \quad , \quad M^{L*}=\left(\frac{G^*(Z^{X*})-G^*(Z^{D*})}{1-G^*(Z^{D*})}\right)M^*=\left(\frac{(Z^{D*})^{-\eta^*}-(Z^{X*})^{-\eta^*}}{(Z^{D*})^{-\eta^*}}\right)M^*$$

$$M^X=\left(\frac{G(Z^I)-G(Z^X)}{1-G(Z^D)}\right)M=\left(\frac{(Z^X)^{-\eta}-(Z^I)^{-\eta}}{(Z^D)^{-\eta}}\right)M \quad , \quad M^{X*}=\left(\frac{G^*(Z^{I*})-G^*(Z^{X*})}{1-G^*(Z^{D*})}\right)M^*=\left(\frac{(Z^{X*})^{-\eta^*}-(Z^{I*})^{-\eta^*}}{(Z^{D*})^{-\eta^*}}\right)M^*$$

$$M^I=\left(\frac{1-G(Z^I)}{1-G(Z^D)}\right)M=\left(\frac{Z^I}{Z^D}\right)^{-\eta}M \quad , \quad M^{I*}=\left(\frac{1-G^*(Z^{I*})}{1-G^*(Z^{D*})}\right)M^*=\left(\frac{Z^{I*}}{Z^{D*}}\right)^{-\eta^*}M^*$$

Home Government Budget Balance (Equation 15):

$$\left(\begin{array}{l} (1-\tau_L)\frac{W}{P_0}L+M(\tau_C-1)\frac{J(Z^D)}{1-G(Z^D)}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}\left(\frac{W}{P_0}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^H\right)^{\theta\sigma+1-\sigma} \\ +M^{X*}(\tau_C-1)\frac{J^*(Z^{X*},Z^{I*})}{G^*(Z^{I*})-G^*(Z^{X*})}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R^*}\right)^{1-\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{W^*}{P_0}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma} \\ +M^{I*}(\tau_C-1)\frac{J^*(Z^{I*})}{1-G^*(Z^{I*})}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}(1-s_V)^{1-\sigma}\left(\frac{W}{P_0}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma} \\ +M(1-\tau_R)\frac{J(Z^D)}{1-G(Z^D)}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}\left(\frac{W}{P_0}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^H\right)^{\theta\sigma+1-\sigma} \\ +M(1-\tau_R)\frac{J(Z^X,Z^I)}{1-G(Z^D)}\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}\left(\frac{W}{P_0^*}\right)^{1-\sigma}(\tau_C^*)^{-\sigma}\left(C^{H*}\right)^{\theta\sigma+1-\sigma} \\ +M^*(1-\tau_R)\left(\frac{J^*(Z^{I*})}{1-G^*(Z^{D*})}\right)\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}(1-s_V)^{1-\sigma}\left(\frac{W}{P_0}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma} \end{array}\right)$$

$$=\left(+\quad M^{I*}\frac{J^*(Z^{I*})}{1-G^*(Z^{I*})}s_V\left(\frac{1}{\rho}\right)^{-\sigma}\left(\frac{1}{\tau_R}\right)^{-\sigma}(1-s_V)^{-\sigma}\left(\frac{W}{P_0}\right)^{-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\right)$$

Nontradable Homogeneous Goods Market Clearing and Household Budget Constraints:

The following two conditions pin down labor used in the homogeneous sector, L_0 and L_0^*.

$$\begin{aligned} L_0 = C_0 &= \tau_L \tfrac{W}{P_0} L - \left(C^H\right)^\theta - \left(C^F\right)^\theta \\ L_0^* = C_0^* &= \tau_L^* \tfrac{W^*}{P_0^*} L^* - \left(C^{F*}\right)^\theta - \left(C^{H*}\right)^\theta \end{aligned}$$

Labor Market Clearing Conditions: By Walras' law, we need only one market clearing condition from the two labor markets in the equilibrium system, and it pins down the world relative wage, $\frac{W^*}{W} = \frac{P_0^*}{P_0}$, which can be interpreted as the real exchange rate in our model economy.

$$L - L_0 = \left(\begin{array}{ll} & M^E F^D \\ + & M^L \int_{Z^D}^{Z^X} \left(l^D(z) + f^D\right) \frac{dG(z)}{G(Z^X) - G(Z^D)} \\ + & M^X \int_{Z^X}^{Z^I} \left(l^{D,X}(z) + f^D\right) \frac{dG(z)}{G(Z^I) - G(Z^X)} \\ + & M^I \int_{Z^I}^{\infty} \left(l^{D,I}(z) + f^D\right) \frac{dG(z)}{1 - G(Z^I)} \\ + & M^X \int_{Z^X}^{Z^I} \left(l^X(z) + f^X\right) \frac{dG(z)}{G(Z^I) - G(Z^X)} \\ + & M^{I*} \int_{Z^{I*}}^{\infty} \left(l^I(z) + f^I\right) \frac{dG^*(z)}{1 - G^*(Z^{I*})} \end{array} \right)$$

$$L^* - L_0^* = \left(\begin{array}{ll} & M^{E*} F^{D*} \\ + & M^{L*} \int_{Z^{D*}}^{Z^{X*}} \left(l^{D*}(z) + f^{D*}\right) \frac{dG^*(z)}{G^*(Z^{X*}) - G^*(Z^{D*})} \\ + & M^{X*} \int_{Z^{X*}}^{Z^{I*}} \left(l^{D,X*}(z) + f^{D*}\right) \frac{dG^*(z)}{G^*(Z^{I*}) - G^*(Z^{X*})} \\ + & M^{I*} \int_{Z^{I*}}^{\infty} \left(l^{D,I*}(z) + f^{D*}\right) \frac{dG^*(z)}{1 - G^*(Z^{I*})} \\ + & M^{X*} \int_{Z^{X*}}^{Z^{I*}} \left(l^{X*}(z) + f^{X*}\right) \frac{dG^*(z)}{G^*(Z^{I*}) - G^*(Z^{X*})} \\ + & M^{I} \int_{Z^{I}}^{\infty} \left(l^{I*}(z) + f^{I*}\right) \frac{dG(z)}{1 - G(Z^{I})} \end{array} \right)$$

Since we conduct the partial-equilibrium analysis, we exclude the labor market clearing condition. The relative wage is assumed to be unity: $\frac{W^*}{W} = \frac{P_0^*}{P_0} = 1$.

2 Characterization of the Equilibrium

This section presents derivations for the analytical solutions for 15 variables: C^H, C^{H*}, C^{F*}, C^F, V, V^*, Z^D, Z^X, Z^I, Z^{D*}, Z^{X*}, Z^{I*}, M, M^* and one tax variable from $\{\tau_L, \tau_C, \tau_R\}$. We take the partial equilibrium analysis: $\frac{W^*}{W} = \frac{P_0^*}{P_0} = 1$ with $\frac{W}{P_0} = \frac{W^*}{P_0^*} = 1$. There is no government in Foreign: $s_V^* = 0$ and $\tau_L^* = \tau_C^* = \tau_R^* = 1$. Given a subsidy rate, the government tax is found numerically from the balanced government budget. Parameters are restricted by

$$\sigma > 1, \quad \tau > 1, \quad \tau^* > 1, \quad \eta > \sigma - 1, \quad \eta^* > \sigma - 1.$$

2.1 Find the equilibrium under exogenous firm mass

We first take M and M^* as given and solve for 13 variables in terms of firm masses.

Find C^H, A^H, Z^D: C^H can be solved out by

$$\begin{aligned}
&\left(C^H\right)^{(\sigma-1)(1-\theta)} \\
&= M\left(\frac{1}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\left(\widetilde{Z}^D\right)^{\sigma-1} \\
&= M\left(\frac{1}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\frac{\eta}{\eta-\sigma+1}\left(Z^D\right)^{\sigma-1} \\
&= M\left(\frac{1}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\frac{\eta}{\eta-\sigma+1}\left(\frac{f^D}{(\tau_R)^\sigma\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}A^H}\right) \\
&= M\left(\frac{1}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\frac{\eta}{\eta-\sigma+1}\left(\frac{f^D}{(\tau_R)^\sigma\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}(\tau_C)^{-\sigma}(C^H)^{\theta\sigma+1-\sigma}}\right) \\
&= M\left(\sigma\frac{\tau_C}{\tau_R}\right)\left(\frac{\eta}{\eta-\sigma+1}\right)\left(\frac{f^D}{(C^H)^{\theta\sigma+1-\sigma}}\right)
\end{aligned}$$

$$\therefore \quad \left(C^H\right)^{\theta\sigma+1-\sigma+(\sigma-1)(1-\theta)} = \left(C^H\right)^\theta = M\left(\sigma\frac{\tau_C}{\tau_R}\right)\left(\frac{\eta}{\eta-\sigma+1}\right)f^D$$

$$i.e. \quad C^H = \left(M\left(\sigma\frac{\tau_C}{\tau_R}\right)\left(\frac{\eta}{\eta-\sigma+1}\right)f^D\right)^{\frac{1}{\theta}}$$

where $\sigma \equiv \frac{1}{1-\rho}$, $1-\sigma = \frac{-\rho}{1-\rho}$, and $\rho = \frac{\sigma-1}{\sigma}$ with $\sigma > 1$ and $0 < \rho < 1$.

Hence, we obtain:

$$A^H = (\tau_C)^{-\sigma}\left(C^H\right)^{\theta\sigma+1-\sigma}$$

$$Z^D = \left(\frac{f^D}{(\tau_R)^\sigma\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}A^H}\right)^{\frac{1}{\sigma-1}}$$

Find C^{F*}, A^{F*}, Z^{D*}: C^{F*} can be solved out by

$$\begin{aligned}
&\left(C^{F*}\right)^{(\sigma-1)(1-\theta)}\\
&= M^*\left(\frac{1}{\rho}\right)^{-(\sigma-1)}\left(\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)}\left(\widetilde{Z}^{D*}\right)^{\sigma-1}\\
&= M^*\left(\frac{1}{\rho}\right)^{-(\sigma-1)}\left(\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)}\frac{\eta^*}{\eta^*-\sigma+1}\left(Z^{D*}\right)^{\sigma-1}\\
&= M^*\left(\frac{1}{\rho}\right)^{-(\sigma-1)}\left(\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)}\frac{\eta^*}{\eta^*-\sigma+1}\left(\frac{f^{D*}}{\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}A^{F*}}\right)\\
&= M^*\left(\frac{1}{\rho}\right)^{-(\sigma-1)}\left(\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)}\frac{\eta^*}{\eta^*-\sigma+1}\left(\frac{f^{D*}}{\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\left(\tau_C^*\right)^{-\sigma}\left(C^{F*}\right)^{\theta\sigma+1-\sigma}}\right)\\
&= M^*\left(\sigma\frac{\tau_C^*}{\tau_R^*}\right)\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{f^{D*}}{\left(C^{F*}\right)^{\theta\sigma+1-\sigma}}\right)
\end{aligned}$$

$$\therefore\quad \left(C^{F*}\right)^{\theta\sigma+1-\sigma+(\sigma-1)(1-\theta)} = \left(C^{F*}\right)^{\theta} = M^*\left(\sigma\frac{\tau_C^*}{\tau_R^*}\right)\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)f^{D*}$$

$$i.e.\quad C^{F*} = \left(M^*\left(\sigma\frac{\tau_C^*}{\tau_R^*}\right)\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)f^{D*}\right)^{\frac{1}{\theta}}$$

where $\sigma \equiv \frac{1}{1-\rho}$, $1-\sigma = \frac{-\rho}{1-\rho}$, and $\rho = \frac{\sigma-1}{\sigma}$ with $\sigma > 1$ and $0 < \rho < 1$.

Hence, we obtain:

$$A^{F*} = \left(\tau_C^*\right)^{-\sigma}\left(C^{F*}\right)^{\theta\sigma+1-\sigma}$$

$$Z^{D*} = \left(\frac{f^{D*}}{\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}A^{F*}}\right)^{\frac{1}{\sigma-1}}$$

Find C^F, A^F, Z^{X*}, Z^{I*}: C^F can be solved out by

$$
\left(C^F\right)^{(\sigma-1)(1-\theta)} = \begin{pmatrix} + & M^{X*}\left(\frac{\tau^*}{\rho}\frac{\tau_C}{\tau_R^*}\right)^{-(\sigma-1)}\left(\widetilde{Z}^{X*}\right)^{\sigma-1} \\ + & M^{I*}\left(\frac{1-s_V}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\left(\widetilde{Z}^{I*}\right)^{\sigma-1} \end{pmatrix}
$$

$$
= \begin{pmatrix} + & M^{*}\left(\frac{\left(Z^{X*}\right)^{-\eta^*}-\left(Z^{I*}\right)^{-\eta^*}}{\left(Z^{D*}\right)^{-\eta^*}}\right)\left(\frac{\tau^*}{\rho}\frac{\tau_C}{\tau_R^*}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{\left(Z^{X*}\right)^{-(\eta^*-\sigma+1)}-\left(Z^{I*}\right)^{-(\eta^*-\sigma+1)}}{\left(Z^{X*}\right)^{-\eta^*}-\left(Z^{I*}\right)^{-\eta^*}}\right) \\ + & M^{*}\left(\frac{Z^{I*}}{Z^{D*}}\right)^{-\eta^*}\left(\frac{1-s_V}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(Z^{I*}\right)^{\sigma-1} \end{pmatrix}
$$

$$
= \begin{pmatrix} + & M^{*}\left(\frac{\tau^*}{\rho}\frac{\tau_C}{\tau_R^*}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{\left(Z^{X*}\right)^{-(\eta^*-\sigma+1)}-\left(Z^{I*}\right)^{-(\eta^*-\sigma+1)}}{\left(Z^{D*}\right)^{-\eta^*}}\right) \\ + & M^{*}\left(\frac{1-s_V}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{\left(Z^{I*}\right)^{-\eta^*+\sigma-1}}{\left(Z^{D*}\right)^{-\eta^*}}\right) \end{pmatrix}
$$

$$
= \begin{pmatrix} + & M^{*}\left(\frac{\tau^*}{\rho}\frac{\tau_C}{\tau_R^*}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right) \\ & \left(\frac{\left(\frac{f^{X*}}{\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}A^F}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}-\left(\frac{f^I-f^{X*}}{\left[\left(\tau_R\right)^{\sigma}\left(\frac{1}{1-s_V}\right)^{\sigma-1}A^F-\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}A^F\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\left(Z^{D*}\right)^{-\eta^*}}\right) \\ + & M^{*}\left(\frac{1-s_V}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\frac{\left(\frac{f^I-f^{X*}}{\left[\left(\tau_R\right)^{\sigma}\left(\frac{1}{1-s_V}\right)^{\sigma-1}A^F-\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}A^F\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\left(Z^{D*}\right)^{-\eta^*}} \end{pmatrix}
$$

$$
\therefore \left(C^F\right)^{\frac{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta^*}{\sigma-1}} =
$$

$$
\begin{pmatrix} + & M^{*}\left(\frac{\tau^*}{\rho}\frac{\tau_C}{\tau_R^*}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right) \\ & \left(\frac{\left(\frac{f^{X*}}{\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\left(\frac{\tau_C}{\tau_R^*}\right)^{-\sigma}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}-\left(\frac{f^I-f^{X*}}{\left[\left(\frac{1}{1-s_V}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R}\right)^{-\sigma}-\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R^*}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\left(Z^{D*}\right)^{-\eta^*}}\right) \\ + & M^{*}\left(\frac{1-s_V}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\frac{\left(\frac{f^I-f^{X*}}{\left[\left(\frac{1}{1-s_V}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R}\right)^{-\sigma}-\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R^*}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\left(Z^{D*}\right)^{-\eta^*}} \end{pmatrix}.
$$

Hence we obtain

$$
A^F = \left(\tau_C\right)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma},
$$

$$
Z^{X*} = \left(\frac{f^{X*}}{\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}A^F}\right)^{\frac{1}{\sigma-1}}, \quad Z^{I*} = \left(\frac{f^I-f^{X*}}{\left[\left(\tau_R\right)^{\sigma}\left(\frac{1}{1-s_V}\right)^{\sigma-1}A^F-\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}A^F\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}.
$$

Find C^{H*}, A^{H*}, Z^X, Z^I:

$$\left(C^{H*}\right)^{(\sigma-1)(1-\theta)} = \left(\begin{array}{ll} + & M^X \left(\frac{\tau}{\rho}\frac{\tau_C^*}{\tau_R}\right)^{-(\sigma-1)} \left(\widetilde{Z}^X\right)^{\sigma-1} \\ + & M^I \left(\frac{1-s_V^*}{\rho}\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)} \left(\widetilde{Z}^I\right)^{\sigma-1} \end{array} \right)$$

$$= \left(\begin{array}{ll} + & M \left(\frac{(Z^X)^{-\eta}-(Z^I)^{-\eta}}{(Z^D)^{-\eta}}\right) \left(\frac{\tau}{\rho}\frac{\tau_C^*}{\tau_R}\right)^{-(\sigma-1)} \left(\frac{\eta}{\eta-\sigma+1}\right) \left(\frac{(Z^X)^{-(\eta-\sigma+1)}-(Z^I)^{-(\eta-\sigma+1)}}{(Z^X)^{-\eta}-(Z^I)^{-\eta}}\right) \\ + & M \left(\frac{Z^I}{Z^D}\right)^{-\eta} \left(\frac{1-s_V^*}{\rho}\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)} \left(\frac{\eta}{\eta-\sigma+1}\right) (Z^I)^{\sigma-1} \end{array} \right)$$

$$= \left(\begin{array}{ll} + & M \left(\frac{\tau}{\rho}\frac{\tau_C^*}{\tau_R}\right)^{-(\sigma-1)} \left(\frac{\eta}{\eta-\sigma+1}\right) \left(\frac{(Z^X)^{-(\eta-\sigma+1)}-(Z^I)^{-(\eta-\sigma+1)}}{(Z^D)^{-\eta}}\right) \\ + & M \left(\frac{1-s_V^*}{\rho}\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)} \left(\frac{\eta}{\eta-\sigma+1}\right) \frac{(Z^I)^{-\eta+\sigma-1}}{(Z^D)^{-\eta}} \end{array} \right)$$

$$= \left(\begin{array}{ll} + & M \left(\frac{\tau}{\rho}\frac{\tau_C^*}{\tau_R}\right)^{-(\sigma-1)} \left(\frac{\eta}{\eta-\sigma+1}\right) \\ & \left(\frac{\left(\frac{f^X}{(\tau_R)^\sigma\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}A^{H*}}\right)^{\frac{-\eta+\sigma-1}{\sigma-1}} - \left(\frac{f^{I*}-f^X}{\left[(\tau_R^*)^\sigma\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}A^{H*}-(\tau_R)^\sigma\left(\frac{1}{\tau}\right)^{\sigma-1}A^{H*}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}}}{(Z^D)^{-\eta}} \right) \\ + & M \left(\frac{1-s_V^*}{\rho}\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)} \left(\frac{\eta}{\eta-\sigma+1}\right) \frac{\left(\frac{f^{I*}-f^X}{\left[(\tau_R^*)^\sigma\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}A^{H*}-(\tau_R)^\sigma\left(\frac{1}{\tau}\right)^{\sigma-1}A^{H*}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}}}{(Z^D)^{-\eta}} \end{array} \right)$$

$$\therefore \left(C^{H*}\right)^{\frac{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta}{\sigma-1}} =$$

$$\left(\begin{array}{ll} + & M \left(\frac{\tau}{\rho}\frac{\tau_C^*}{\tau_R}\right)^{-(\sigma-1)} \left(\frac{\eta}{\eta-\sigma+1}\right) \\ & \left(\frac{\left(\frac{f^X}{\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R}\right)^{-\sigma}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}} - \left(\frac{f^{I*}-f^X}{\left[\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R^*}\right)^{-\sigma}-\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}}}{(Z^D)^{-\eta}} \right) \\ + & M \left(\frac{1-s_V^*}{\rho}\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)} \left(\frac{\eta}{\eta-\sigma+1}\right) \frac{\left(\frac{f^{I*}-f^X}{\left[\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R^*}\right)^{-\sigma}-\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}}}{(Z^D)^{-\eta}} \end{array} \right)$$

In turn, we obtain

$$A^{H*} = \left(\tau_C^*\right)^{-\sigma} \left(C^{H*}\right)^{\theta\sigma+1-\sigma},$$

$$Z^X = \left(\frac{f^X}{(\tau_R)^\sigma\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}A^{H*}}\right)^{\frac{1}{\sigma-1}}, \quad Z^I = \left(\frac{f^{I*}-f^X}{\left[(\tau_R^*)^\sigma\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}A^{H*}-(\tau_R)^\sigma\left(\frac{1}{\tau}\right)^{\sigma-1}A^{H*}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}.$$

Therefore, given M and M^*, and given solutions for C^H, C^{H*}, C^{F*}, C^F, Z^D, Z^X, Z^I, Z^{D*}, Z^{X*}, and Z^{I*}, we can find all the other endogenous variables: consumption of homogeneous goods, masses of entrants, locals, exporters, and multinationals, taxes, and welfare.

Mass of Firms:

$$M^E = \frac{\delta M}{1-G(Z^D)} \quad , \quad M^{E*} = \frac{\delta M^*}{1-G^*(Z^{D*})}$$

$$M^L = \left(\frac{G(Z^X)-G(Z^D)}{1-G(Z^D)}\right)M = \left(\frac{(Z^D)^{-\eta}-(Z^X)^{-\eta}}{(Z^D)^{-\eta}}\right)M \quad , \quad M^{L*} = \left(\frac{G^*(Z^{X*})-G^*(Z^{D*})}{1-G^*(Z^{D*})}\right)M^* = \left(\frac{(Z^{D*})^{-\eta^*}-(Z^{X*})^{-\eta^*}}{(Z^{D*})^{-\eta^*}}\right)M^*$$

$$M^X = \left(\frac{G(Z^I)-G(Z^X)}{1-G(Z^D)}\right)M = \left(\frac{(Z^X)^{-\eta}-(Z^I)^{-\eta}}{(Z^D)^{-\eta}}\right)M \quad , \quad M^{X*} = \left(\frac{G^*(Z^{I*})-G^*(Z^{X*})}{1-G^*(Z^{D*})}\right)M^* = \left(\frac{(Z^{X*})^{-\eta^*}-(Z^{I*})^{-\eta^*}}{(Z^{D*})^{-\eta^*}}\right)M^*$$

$$M^I = \left(\frac{1-G(Z^I)}{1-G(Z^D)}\right)M = \left(\frac{Z^I}{Z^D}\right)^{-\eta}M \quad , \quad M^{I*} = \left(\frac{1-G^*(Z^{I*})}{1-G^*(Z^{D*})}\right)M^* = \left(\frac{Z^{I*}}{Z^{D*}}\right)^{-\eta^*}M^*$$

Indirect Utility:

$$V = L\tau_L + \left(\tfrac{1}{\theta}-1\right)\left(C^H\right)^\theta + \left(\tfrac{1}{\theta}-1\right)\left(C^F\right)^\theta$$

$$V^* = L^*\tau_L^* + \left(\tfrac{1}{\theta}-1\right)\left(C^{F*}\right)^\theta + \left(\tfrac{1}{\theta}-1\right)\left(C^{H*}\right)^\theta$$

Consumption on Nontradable Homogeneous Goods:

$$L_0 = C_0 = \tau_L L - \left(C^H\right)^\theta - \left(C^F\right)^\theta$$

$$L_0^* = C_0^* = \tau_L^* L^* - \left(C^{F*}\right)^\theta - \left(C^{H*}\right)^\theta$$

Home Government Budget Balance: Tax variables are numerically pinned down by the government budget balance given a FDI subsidy rate. The government budget balance is given by

$$\begin{aligned}&\left(\begin{array}{l}(1-\tau_L)L+M(\tau_C-1)\frac{J(Z^D)}{1-G(Z^D)}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^H\right)^{\theta\sigma+1-\sigma}\\ +M^{X*}(\tau_C-1)\frac{J^*(Z^{X*},Z^{I*})}{G^*(Z^{I*})-G^*(Z^{X*})}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R^*}\right)^{1-\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\\ +M^{I*}(\tau_C-1)\frac{J^*(Z^{I*})}{1-G^*(Z^{I*})}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}(1-s_V)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\\ +M(1-\tau_R)\frac{J(Z^D)}{1-G(Z^D)}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^H\right)^{\theta\sigma+1-\sigma}\\ +M(1-\tau_R)\frac{J(Z^X,Z^I)}{1-G(Z^D)}\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}(\tau_C^*)^{-\sigma}\left(C^{H*}\right)^{\theta\sigma+1-\sigma}\\ +M^*(1-\tau_R)\left(\frac{J^*(Z^{I*})}{1-G^*(Z^{D*})}\right)\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}(1-s_V)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\end{array}\right)\\ =&\left(+\quad M^{I*}\frac{J^*(Z^{I*})}{1-G^*(Z^{I*})}s_V\left(\frac{1}{\rho}\right)^{-\sigma}\left(\frac{1}{\tau_R}\right)^{-\sigma}(1-s_V)^{-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\right)\end{aligned}$$

Lump-Sum Tax: If the government imposes tax on labor income, then we can pin down τ_L algebraically.

$$L(1-\tau_L)\quad=\left(+\quad M^{I*}\frac{J^*(Z^{I*})}{1-G^*(Z^{I*})}s_V\left(\frac{1}{\rho}\right)^{-\sigma}\left(\frac{1}{\tau_R}\right)^{-\sigma}(1-s_V)^{-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\right)$$

Consumption Tax: Suppose the government imposes consumption tax. Then τ_C can be found numerically by solving

$$\begin{aligned}&\left(\begin{array}{l}+M(\tau_C-1)\frac{J(Z^D)}{1-G(Z^D)}\left(\frac{1}{\rho}\frac{1}{\tau_R}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^H\right)^{\theta\sigma+1-\sigma}\\ +M^{X*}(\tau_C-1)\frac{J^*(Z^{X*},Z^{I*})}{G^*(Z^{I*})-G^*(Z^{X*})}\left(\frac{\tau^*}{\rho}\frac{1}{\tau_R^*}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\\ +M^{I*}(\tau_C-1)\frac{J^*(Z^{I*})}{1-G^*(Z^{I*})}\left(\frac{1-s_V}{\rho}\frac{1}{\tau_R}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\end{array}\right)\\ =&\left(+\quad M^{I*}\frac{J^*(Z^{I*})}{1-G^*(Z^{I*})}s_V\left(\frac{1}{\rho}\right)^{-\sigma}\left(\frac{1}{\tau_R}\right)^{-\sigma}(1-s_V)^{-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\right)\end{aligned}$$

Tax on Firm Revenue: Suppose the government imposes firm-revenue tax. Then τ_R can be found numerically by solving

$$\begin{aligned}&\left(\begin{array}{l}+M(1-\tau_R)\frac{J(Z^D)}{1-G(Z^D)}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^H\right)^{\theta\sigma+1-\sigma}\\ +M(1-\tau_R)\frac{J(Z^X,Z^I)}{1-G(Z^D)}\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}(\tau_C^*)^{-\sigma}\left(C^{H*}\right)^{\theta\sigma+1-\sigma}\\ +M^*(1-\tau_R)\left(\frac{J^*(Z^{I*})}{1-G^*(Z^{D*})}\right)\left(\frac{1}{\rho}\right)^{1-\sigma}\left(\frac{1}{\tau_R}\right)^{1-\sigma}(1-s_V)^{1-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\end{array}\right)\\ =&\left(+\quad M^{I*}\frac{J^*(Z^{I*})}{1-G^*(Z^{I*})}s_V\left(\frac{1}{\rho}\right)^{-\sigma}\left(\frac{1}{\tau_R}\right)^{-\sigma}(1-s_V)^{-\sigma}(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\right)\end{aligned}$$

2.2 Find the equilibrium under endogenous firm mass

Now we endogenize masses of firms and close the equilibrium.

Free Entries: When we endogenize total firm masses, M and M^*, they can be found from free entry conditions, given by

$$\delta F^D = \begin{pmatrix} + & J\left(Z^D\right)(\tau_R)^\sigma\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\left[(\tau_C)^{-\sigma}\left(C^H\right)^{\theta\sigma+1-\sigma}\right] \\ + & J\left(Z^X, Z^I\right)(\tau_R)^\sigma\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{\tau}\right)^{\sigma-1}\left[(\tau_C^*)^{-\sigma}\left(C^{H*}\right)^{\theta\sigma+1-\sigma}\right] \\ + & J\left(Z^I\right)(\tau_R^*)^\sigma\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{(1-s_V^*)}\right)^{\sigma-1}\left[(\tau_C^*)^{-\sigma}\left(C^{H*}\right)^{\theta\sigma+1-\sigma}\right] \\ - & f^D\left(1-G(Z^D)\right) \\ - & f^X\left(G(Z^I)-G(Z^X)\right) \\ - & f^{I*}\left(1-G(Z^I)\right) \end{pmatrix}$$

where

$$J(Z^D) = \frac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left(Z^D\right)^{-\eta+\sigma-1},$$

$$J(Z^I) = \frac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left(Z^I\right)^{-\eta+\sigma-1}, \qquad J(Z^X, Z^I) = \frac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left[\left(Z^X\right)^{-(\eta-\sigma+1)} - \left(Z^I\right)^{-(\eta-\sigma+1)}\right],$$

$$\left(1-G(Z^D)\right) = (z_{min})^\eta\left(Z^D\right)^{-\eta},$$

$$\left(1-G(Z^I)\right) = (z_{min})^\eta\left(Z^I\right)^{-\eta}, \qquad \left(G(Z^I)-G(Z^X)\right) = (z_{min})^\eta\left[\left(Z^X\right)^{-\eta} - \left(Z^I\right)^{-\eta}\right].$$

$$\delta F^{D*} = \begin{pmatrix} + & J^*\left(Z^{D*}\right)(\tau_R^*)^\sigma\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\left[(\tau_C^*)^{-\sigma}\left(C^{F*}\right)^{\theta\sigma+1-\sigma}\right] \\ + & J^*\left(Z^{X*}, Z^{I*}\right)(\tau_R^*)^\sigma\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{\tau^*}\right)^{\sigma-1}\left[(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\right] \\ + & J^*\left(Z^{I*}\right)(\tau_R)^\sigma\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{(1-s_V)}\right)^{\sigma-1}\left[(\tau_C)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma}\right] \\ - & f^{D*}\left(1-G^*(Z^{D*})\right) \\ - & f^{X*}\left(G^*(Z^{I*})-G^*(Z^{X*})\right) \\ - & f^I\left(1-G^*(Z^{I*})\right) \end{pmatrix}$$

where

$$J^*(Z^{D*}) = \frac{\eta^*(z^*_{min})^{\eta^*}}{\eta^*-\sigma+1}\left(Z^{D*}\right)^{-\eta^*+\sigma-1},$$

$$J^*(Z^{I*}) = \frac{\eta^*(z^*_{min})^{\eta^*}}{\eta^*-\sigma+1}\left(Z^{I*}\right)^{-\eta^*+\sigma-1}, \qquad J^*(Z^{X*}, Z^{I*}) = \frac{\eta^*(z^*_{min})^{\eta^*}}{\eta^*-\sigma+1}\left[\left(Z^{X*}\right)^{-(\eta^*-\sigma+1)} - \left(Z^{I*}\right)^{-(\eta^*-\sigma+1)}\right],$$

$$\left(1 - G^*(Z^{D*})\right) = \left(z^*_{min}\right)^{\eta^*}\left(Z^{D*}\right)^{-\eta^*},$$

$$\left(1 - G^*(Z^{I*})\right) = \left(z^*_{min}\right)^{\eta^*}\left(Z^{I*}\right)^{-\eta^*}, \qquad \left(G^*(Z^{I*}) - G^*(Z^{X*})\right) = \left(z^*_{min}\right)^{\eta^*}\left[\left(Z^{X*}\right)^{-\eta^*} - \left(Z^{I*}\right)^{-\eta^*}\right].$$

For Home consumption, demand, cutoff in Home, we can obtain

$$\begin{aligned}
C^H &= \left(\left(\sigma\frac{\tau_C}{\tau_R}\right)\left(\frac{\eta}{\eta-\sigma+1}\right)f^D\right)^{\frac{1}{\theta}}(M)^{\frac{1}{\theta}} \\
A^H &= (\tau_C)^{-\sigma}\left(\left(\sigma\frac{\tau_C}{\tau_R}\right)\left(\frac{\eta}{\eta-\sigma+1}\right)f^D\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}}(M)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
&= \Xi_{A^H}(M)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
Z^D &= \left(\frac{f^D}{(\tau_R)^{\sigma}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}\left(A^H\right)^{\frac{-1}{\sigma-1}} \\
&= \left(\frac{f^D}{(\tau_R)^{\sigma}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}\left(\Xi_{A^H}\right)^{\frac{-1}{\sigma-1}}(M)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}} \\
&= \Xi_{Z^D}(M)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}}
\end{aligned}$$

where $\theta\sigma + 1 - \sigma < 0$ and $\sigma > 1$.

For Foreign consumption, demand, cutoff in Foreign, we can obtain

$$\begin{aligned}
C^{F*} &= \left(\left(\sigma\frac{\tau^*_C}{\tau^*_R}\right)\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)f^{D*}\right)^{\frac{1}{\theta}}(M^*)^{\frac{1}{\theta}} \\
A^{F*} &= (\tau^*_C)^{-\sigma}\left(\left(\sigma\frac{\tau^*_C}{\tau^*_R}\right)\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)f^{D*}\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}}(M^*)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
&= \Xi_{A^{F*}}(M^*)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
Z^{D*} &= \left(\frac{f^{D*}}{(\tau^*_R)^{\sigma}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}\left(A^{F*}\right)^{\frac{-1}{\sigma-1}} \\
&= \left(\frac{f^{D*}}{(\tau^*_R)^{\sigma}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}\left(\Xi_{A^{F*}}\right)^{\frac{-1}{\sigma-1}}(M^*)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}} \\
&= \Xi_{Z^{D*}}(M^*)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}}
\end{aligned}$$

where $\theta\sigma + 1 - \sigma < 0$ and $\sigma > 1$.

For Foreign consumption, demand, cutoff in Home, we can obtain

$$
\begin{aligned}
&\left(C^F\right)^{\frac{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta^*}{\sigma-1}} \\
&= \left(\begin{array}{l}
+ \quad M^*\left(\frac{\tau^*}{\rho}\frac{\tau_C}{\tau_R^*}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right) \\
\left(\frac{\left(\frac{f^{X*}}{\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\left(\frac{\tau_C}{\tau_R^*}\right)^{-\sigma}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}} - \left(\frac{f^I-f^{X*}}{\left[\left(\frac{1}{1-s_V}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R}\right)^{-\sigma}-\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R^*}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\left(Z^{D*}\right)^{-\eta^*}}\right) \\
+ \quad M^*\left(\frac{1-s_V}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\frac{\left(\frac{f^I-f^{X*}}{\left[\left(\frac{1}{1-s_V}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R}\right)^{-\sigma}-\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R^*}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\left(Z^{D*}\right)^{-\eta^*}}
\end{array}\right) \\
&= M^*\left(Z^{D*}\right)^{\eta^*}\left(\begin{array}{l}
+ \quad \left(\frac{\tau^*}{\rho}\frac{\tau_C}{\tau_R^*}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{f^{X*}}{\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\left(\frac{\tau_C}{\tau_R^*}\right)^{-\sigma}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}} \\
- \quad \left(\frac{\tau^*}{\rho}\frac{\tau_C}{\tau_R^*}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{f^I-f^{X*}}{\left[\left(\frac{1}{1-s_V}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R}\right)^{-\sigma}-\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R^*}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}} \\
+ \quad \left(\frac{1-s_V}{\rho}\frac{\tau_C}{\tau_R}\right)^{-(\sigma-1)}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{f^I-f^{X*}}{\left[\left(\frac{1}{1-s_V}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R}\right)^{-\sigma}-\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{\tau_C}{\tau_R^*}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}
\end{array}\right) \\
&= M^*\left(Z^{D*}\right)^{\eta^*}\Xi_{C^F} = M^*\left(\Xi_{Z^{D*}}\right)^{\eta^*}\left(M^*\right)^{\frac{-(\theta\sigma+1-\sigma)\eta^*}{\theta(\sigma-1)}}\Xi_{C^F} = \Xi_{C^F}\left(\Xi_{Z^{D*}}\right)^{\eta^*}\left(M^*\right)^{\frac{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta^*}{\theta(\sigma-1)}}
\end{aligned}
$$

$$
\begin{aligned}
\therefore C^F &= \left(\Xi_{C^F}\left(\Xi_{Z^{D*}}\right)^{\eta^*}\left(M^*\right)^{\frac{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta^*}{\theta(\sigma-1)}}\right)^{\frac{\sigma-1}{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta^*}} = \left[\Xi_{C^F}\left(\Xi_{Z^{D*}}\right)^{\eta^*}\right]^{\frac{\sigma-1}{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta^*}}\left(M^*\right)^{\frac{1}{\theta}} \\
A^F &= \left(\tau_C\right)^{-\sigma}\left(C^F\right)^{\theta\sigma+1-\sigma} = \left(\tau_C\right)^{-\sigma}\left[\Xi_{C^F}\left(\Xi_{Z^{D*}}\right)^{\eta^*}\right]^{\frac{(\sigma-1)(\theta\sigma+1-\sigma)}{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta^*}}\left(M^*\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} = \Xi_{A^F}\left(M^*\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
Z^{X*} &= \left(\frac{f^{X*}}{\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}\left(A^F\right)^{\frac{-1}{\sigma-1}} \\
&= \left(\frac{f^{X*}}{\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}\left(\Xi_{A^F}\right)^{\frac{-1}{\sigma-1}}\left(M^*\right)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}} = \Xi_{Z^{X*}}\left(M^*\right)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}} \\
Z^{I*} &= \left(\frac{f^I-f^{X*}}{\left[\left(\tau_R\right)^{\sigma}\left(\frac{1}{1-s_V}\right)^{\sigma-1}A^F-\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}A^F\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}} \\
&= \left(\frac{f^I-f^{X*}}{\left[\left(\tau_R\right)^{\sigma}\left(\frac{1}{1-s_V}\right)^{\sigma-1}\Xi_{A^F}-\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{\tau^*}\right)^{\sigma-1}\Xi_{A^F}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}\left(M^*\right)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}} = \Xi_{Z^{I*}}\left(M^*\right)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}}
\end{aligned}
$$

where $\theta\sigma+1-\sigma<0$ and $\sigma>1$.

For Home consumption, demand, cutoff in Foreign, we can obtain

$$
\begin{aligned}
&\left(C^{H*}\right)^{\frac{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta}{\sigma-1}} \\
&= \left(\begin{array}{l}
+\quad M\left(\frac{\tau}{\rho}\frac{\tau_C^*}{\tau_R}\right)^{-(\sigma-1)}\left(\frac{\eta}{\eta-\sigma+1}\right) \\
\left(\frac{\left(\frac{f^X}{\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R}\right)^{-\sigma}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}} - \left(\frac{f^{I*}-f^X}{\left[\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R^*}\right)^{-\sigma}-\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}}}{\left(Z^D\right)^{-\eta}}\right) \\
+\quad M\left(\frac{1-s_V^*}{\rho}\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)}\left(\frac{\eta}{\eta-\sigma+1}\right)\frac{\left(\frac{f^{I*}-f^X}{\left[\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R^*}\right)^{-\sigma}-\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}}}{\left(Z^D\right)^{-\eta}}
\end{array}\right) \\
&= M\left(Z^D\right)^{\eta}\left(\begin{array}{l}
+\quad \left(\frac{\tau}{\rho}\frac{\tau_C^*}{\tau_R}\right)^{-(\sigma-1)}\left(\frac{\eta}{\eta-\sigma+1}\right)\left(\frac{f^X}{\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R}\right)^{-\sigma}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}} \\
-\quad \left(\frac{\tau}{\rho}\frac{\tau_C^*}{\tau_R}\right)^{-(\sigma-1)}\left(\frac{\eta}{\eta-\sigma+1}\right)\left(\frac{f^{I*}-f^X}{\left[\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R^*}\right)^{-\sigma}-\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}} \\
+\quad \left(\frac{1-s_V^*}{\rho}\frac{\tau_C^*}{\tau_R^*}\right)^{-(\sigma-1)}\left(\frac{\eta}{\eta-\sigma+1}\right)\left(\frac{f^{I*}-f^X}{\left[\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R^*}\right)^{-\sigma}-\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{\tau_C^*}{\tau_R}\right)^{-\sigma}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-(\eta-\sigma+1)}{\sigma-1}}
\end{array}\right) \\
&= M\left(Z^D\right)^{\eta}\Xi_{C^{H*}} = M\left(\Xi_{Z^D}\right)^{\eta}(M)^{\frac{-(\theta\sigma+1-\sigma)\eta}{\theta(\sigma-1)}}\Xi_{C^{H*}} = \Xi_{C^{H*}}\left(\Xi_{Z^D}\right)^{\eta}(M)^{\frac{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta}{\theta(\sigma-1)}}
\end{aligned}
$$

$$
\begin{aligned}
\therefore C^{H*} &= \left(\Xi_{C^{H*}}\left(\Xi_{Z^D}\right)^{\eta}(M)^{\frac{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta}{\theta(\sigma-1)}}\right)^{\frac{\sigma-1}{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta}} = \left[\Xi_{C^{H*}}\left(\Xi_{Z^D}\right)^{\eta}\right]^{\frac{\sigma-1}{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta}}(M)^{\frac{1}{\theta}} \\
A^{H*} &= \left(\tau_C^*\right)^{-\sigma}\left(C^{H*}\right)^{\theta\sigma+1-\sigma} = \left(\tau_C^*\right)^{-\sigma}\left[\Xi_{C^{H*}}\left(\Xi_{Z^D}\right)^{\eta}\right]^{\frac{(\sigma-1)(\theta\sigma+1-\sigma)}{\theta(\sigma-1)-(\theta\sigma+1-\sigma)\eta}}(M)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} = \Xi_{A^{H*}}(M)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
Z^X &= \left(\frac{f^X}{(\tau_R)^{\sigma}\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}\left(A^{H*}\right)^{\frac{-1}{\sigma-1}} \\
&= \left(\frac{f^X}{(\tau_R)^{\sigma}\left(\frac{1}{\tau}\right)^{\sigma-1}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}\left(\Xi_{A^{H*}}\right)^{\frac{-1}{\sigma-1}}(M)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}} = \Xi_{Z^X}(M)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}} \\
Z^I &= \left(\frac{f^{I*}-f^X}{\left[\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}A^{H*}-(\tau_R)^{\sigma}\left(\frac{1}{\tau}\right)^{\sigma-1}A^{H*}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}} \\
&= \left(\frac{f^{I*}-f^X}{\left[\left(\tau_R^*\right)^{\sigma}\left(\frac{1}{1-s_V^*}\right)^{\sigma-1}\Xi_{A^{H*}}-(\tau_R)^{\sigma}\left(\frac{1}{\tau}\right)^{\sigma-1}\Xi_{A^{H*}}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}(M)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}} = \Xi_{Z^I}(M)^{\frac{-(\theta\sigma+1-\sigma)}{\theta(\sigma-1)}}
\end{aligned}
$$

where $\theta\sigma+1-\sigma<0$ and $\sigma>1$.

Therefore, observe that

$$\begin{aligned}
J(Z^D)A^H &= \tfrac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left(Z^D\right)^{-\eta+\sigma-1}\Xi_{A^H}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
&= \tfrac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left(\Xi_{Z^D}\right)^{-\eta+\sigma-1}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)(-\eta+\sigma-1)}{\theta(1-\sigma)}}\Xi_{A^H}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
&= \tfrac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left(\Xi_{Z^D}\right)^{-\eta+\sigma-1}\Xi_{A^H}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)\eta}{\theta(\sigma-1)}}
\end{aligned}$$

$$\begin{aligned}
J(Z^X,Z^I)A^{H*} &= \tfrac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left[\left(Z^X\right)^{-(\eta-\sigma+1)}-\left(Z^I\right)^{-(\eta-\sigma+1)}\right]\Xi_{A^{H*}}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
&= \tfrac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left[\left(\Xi_{Z^X}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta(1-\sigma)}}\right)^{-(\eta-\sigma+1)}-\left(\Xi_{Z^I}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta(1-\sigma)}}\right)^{-(\eta-\sigma+1)}\right]\Xi_{A^{H*}}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
&= \tfrac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left[\left(\Xi_{Z^X}\right)^{-(\eta-\sigma+1)}-\left(\Xi_{Z^I}\right)^{-(\eta-\sigma+1)}\right]\Xi_{A^{H*}}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)\eta}{\theta(\sigma-1)}}
\end{aligned}$$

$$\begin{aligned}
J(Z^I)A^{H*} &= \tfrac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left(Z^I\right)^{-\eta+\sigma-1}\Xi_{A^{H*}}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
&= \tfrac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left(\Xi_{Z^I}\right)^{-\eta+\sigma-1}\Xi_{A^{H*}}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)\eta}{\theta(\sigma-1)}}
\end{aligned}$$

$$\begin{aligned}
\left(1-G(Z^D)\right) &= (z_{min})^\eta\left(Z^D\right)^{-\eta} &&= (z_{min})^\eta\left(\Xi_{Z^D}\right)^{-\eta}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)\eta}{\theta(\sigma-1)}} \\
\left(1-G(Z^I)\right) &= (z_{min})^\eta\left(Z^I\right)^{-\eta} &&= (z_{min})^\eta\left(\Xi_{Z^I}\right)^{-\eta}\left(M\right)^{\frac{(\theta\sigma+1-\sigma)\eta}{\theta(\sigma-1)}} \\
\left(G(Z^I)-G(Z^X)\right) &= (z_{min})^\eta\left[\left(Z^X\right)^{-\eta}-\left(Z^I\right)^{-\eta}\right] &&= (z_{min})^\eta\left[\left(\Xi_{Z^X}\right)^{-\eta}-\left(\Xi_{Z^I}\right)^{-\eta}\right]\left(M\right)^{\frac{(\theta\sigma+1-\sigma)\eta}{\theta(\sigma-1)}}
\end{aligned}$$

Hence, the free entry condition for Home firms can be rewritten as

$$\delta F^D = \left(M\right)^{\frac{(\theta\sigma+1-\sigma)\eta}{\theta(\sigma-1)}}\left(\begin{array}{ll}
+ & \left(\tau_R\right)^\sigma\left(\frac{1}{\sigma}\right)\left(\rho\right)^{\sigma-1}\,\frac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left(\Xi_{Z^D}\right)^{-\eta+\sigma-1}\Xi_{A^H} \\
+ & \left(\tau_R\right)^\sigma\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{\tau}\right)^{\sigma-1}\,\frac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left[\left(\Xi_{Z^X}\right)^{-(\eta-\sigma+1)}-\left(\Xi_{Z^I}\right)^{-(\eta-\sigma+1)}\right]\Xi_{A^{H*}} \\
+ & \left(\tau_R^*\right)^\sigma\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{(1-s_V^*)}\right)^{\sigma-1}\,\frac{\eta(z_{min})^\eta}{\eta-\sigma+1}\left(\Xi_{Z^I}\right)^{-\eta+\sigma-1}\Xi_{A^{H*}} \\
- & f^D\left(z_{min}\right)^\eta\left(\Xi_{Z^D}\right)^{-\eta} \\
- & f^X\left(z_{min}\right)^\eta\left[\left(\Xi_{Z^X}\right)^{-\eta}-\left(\Xi_{Z^I}\right)^{-\eta}\right] \\
- & f^{I*}\left(z_{min}\right)^\eta\left(\Xi_{Z^I}\right)^{-\eta}
\end{array}\right)$$

For the counterpart for Foreign, we have

$$\begin{aligned}
J^{*}\left(Z^{D*}\right)A^{F*} &= \tfrac{\eta^{*}\left(z^{*}_{min}\right)^{\eta^{*}}}{\eta^{*}-\sigma+1}\left(Z^{D*}\right)^{-\eta^{*}+\sigma-1}\Xi_{A^{F*}}\left(M^{*}\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
&= \tfrac{\eta^{*}\left(z^{*}_{min}\right)^{\eta^{*}}}{\eta^{*}-\sigma+1}\left(\Xi_{Z^{D*}}\right)^{-\eta^{*}+\sigma-1}\Xi_{A^{F*}}\left(M^{*}\right)^{\frac{(\theta\sigma+1-\sigma)\eta^{*}}{\theta(\sigma-1)}} \\
J^{*}\left(Z^{X*},Z^{I*}\right)A^{F} &= \tfrac{\eta^{*}\left(z^{*}_{min}\right)^{\eta^{*}}}{\eta^{*}-\sigma+1}\left[\left(Z^{X*}\right)^{-(\eta^{*}-\sigma+1)}-\left(Z^{I*}\right)^{-(\eta^{*}-\sigma+1)}\right]\Xi_{A^{F}}\left(M^{*}\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
&= \tfrac{\eta^{*}\left(z^{*}_{min}\right)^{\eta^{*}}}{\eta^{*}-\sigma+1}\left[\left(\Xi_{Z^{X*}}\right)^{-(\eta^{*}-\sigma+1)}-\left(\Xi_{Z^{I*}}\right)^{-(\eta^{*}-\sigma+1)}\right]\Xi_{A^{F}}\left(M^{*}\right)^{\frac{(\theta\sigma+1-\sigma)\eta^{*}}{\theta(\sigma-1)}} \\
J^{*}(Z^{I*})A^{F} &= \tfrac{\eta^{*}\left(z^{*}_{min}\right)^{\eta^{*}}}{\eta^{*}-\sigma+1}\left(Z^{I*}\right)^{-\eta^{*}+\sigma-1}\Xi_{A^{F}}\left(M^{*}\right)^{\frac{(\theta\sigma+1-\sigma)}{\theta}} \\
&= \tfrac{\eta^{*}\left(z^{*}_{min}\right)^{\eta^{*}}}{\eta^{*}-\sigma+1}\left(\Xi_{Z^{I*}}\right)^{-\eta^{*}+\sigma-1}\Xi_{A^{F}}\left(M^{*}\right)^{\frac{(\theta\sigma+1-\sigma)\eta^{*}}{\theta(\sigma-1)}}
\end{aligned}$$

$$\begin{aligned}
\left(1-G^{*}(Z^{D*})\right) &= \left(z^{*}_{min}\right)^{\eta^{*}}\left(Z^{D*}\right)^{-\eta^{*}} && = \left(z^{*}_{min}\right)^{\eta^{*}}\left(\Xi_{Z^{D*}}\right)^{-\eta^{*}}\left(M^{*}\right)^{\frac{(\theta\sigma+1-\sigma)\eta^{*}}{\theta(\sigma-1)}} \\
\left(1-G^{*}(Z^{I*})\right) &= \left(z^{*}_{min}\right)^{\eta^{*}}\left(Z^{I*}\right)^{-\eta^{*}} && = \left(z^{*}_{min}\right)^{\eta^{*}}\left(\Xi_{Z^{I*}}\right)^{-\eta^{*}}\left(M^{*}\right)^{\frac{(\theta\sigma+1-\sigma)\eta^{*}}{\theta(\sigma-1)}} \\
\left(G^{*}(Z^{I*})-G^{*}(Z^{X*})\right) &= \left(z^{*}_{min}\right)^{\eta^{*}}\left[\left(Z^{X*}\right)^{-\eta^{*}}-\left(Z^{I*}\right)^{-\eta^{*}}\right] && = \left(z^{*}_{min}\right)^{\eta^{*}}\left[\left(\Xi_{Z^{X*}}\right)^{-\eta^{*}}-\left(\Xi_{Z^{I*}}\right)^{-\eta^{*}}\right]\left(M^{*}\right)^{\frac{(\theta\sigma+1-\sigma)\eta^{*}}{\theta(\sigma-1)}}
\end{aligned}$$

Hence, the free entry condition for Foreign firms can be rewritten as

$$\delta F^{D*} = \left(M^{*}\right)^{\frac{(\theta\sigma+1-\sigma)\eta^{*}}{\theta(\sigma-1)}}\left(\begin{array}{ll}
+ & \left(\tau^{*}_{R}\right)^{\sigma}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\,\frac{\eta^{*}\left(z^{*}_{min}\right)^{\eta^{*}}}{\eta^{*}-\sigma+1}\left(\Xi_{Z^{D*}}\right)^{-\eta^{*}+\sigma-1}\Xi_{A^{F*}} \\
+ & \left(\tau^{*}_{R}\right)^{\sigma}\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{\tau^{*}}\right)^{\sigma-1}\,\frac{\eta^{*}\left(z^{*}_{min}\right)^{\eta^{*}}}{\eta^{*}-\sigma+1}\left[\left(\Xi_{Z^{X*}}\right)^{-(\eta^{*}-\sigma+1)}-\left(\Xi_{Z^{I*}}\right)^{-(\eta^{*}-\sigma+1)}\right]\Xi_{A^{F}} \\
+ & \left(\tau_{R}\right)^{\sigma}\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{(1-s_{V})}\right)^{\sigma-1}\,\frac{\eta^{*}\left(z^{*}_{min}\right)^{\eta^{*}}}{\eta^{*}-\sigma+1}\left(\Xi_{Z^{I*}}\right)^{-\eta^{*}+\sigma-1}\Xi_{A^{F}} \\
- & f^{D*}\left(z^{*}_{min}\right)^{\eta^{*}}\left(\Xi_{Z^{D*}}\right)^{-\eta^{*}} \\
- & f^{X*}\left(z^{*}_{min}\right)^{\eta^{*}}\left[\left(\Xi_{Z^{X*}}\right)^{-\eta^{*}}-\left(\Xi_{Z^{I*}}\right)^{-\eta^{*}}\right] \\
- & f^{I}\left(z^{*}_{min}\right)^{\eta^{*}}\left(\Xi_{Z^{I*}}\right)^{-\eta^{*}}
\end{array}\right)$$

In sum, the two free entry conditions pin down M and M^{*}:

$$(M)^{\frac{(\sigma-1-\theta\sigma)\eta}{\theta(\sigma-1)}} = \frac{1}{\delta F^{D}} \left(\begin{array}{ll} + & (\tau_R)^{\sigma}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\frac{\eta(z_{min})^{\eta}}{\eta-\sigma+1}\left(\Xi_{Z^{D}}\right)^{-\eta+\sigma-1}\Xi_{A^{H}} \\ + & (\tau_R)^{\sigma}\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{\tau}\right)^{\sigma-1}\frac{\eta(z_{min})^{\eta}}{\eta-\sigma+1}\left[\left(\Xi_{Z^{X}}\right)^{-(\eta-\sigma+1)}-\left(\Xi_{Z^{I}}\right)^{-(\eta-\sigma+1)}\right]\Xi_{A^{H*}} \\ + & (\tau_R^{*})^{\sigma}\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{(1-s_V^{*})}\right)^{\sigma-1}\frac{\eta(z_{min})^{\eta}}{\eta-\sigma+1}\left(\Xi_{Z^{I}}\right)^{-\eta+\sigma-1}\Xi_{A^{H*}} \\ - & f^{D}\left(z_{min}\right)^{\eta}\left(\Xi_{Z^{D}}\right)^{-\eta} \\ - & f^{X}\left(z_{min}\right)^{\eta}\left[\left(\Xi_{Z^{X}}\right)^{-\eta}-\left(\Xi_{Z^{I}}\right)^{-\eta}\right] \\ - & f^{I*}\left(z_{min}\right)^{\eta}\left(\Xi_{Z^{I}}\right)^{-\eta} \end{array} \right)$$

$$(M^{*})^{\frac{(\sigma-1-\theta\sigma)\eta^{*}}{\theta(\sigma-1)}} = \frac{1}{\delta F^{D*}} \left(\begin{array}{ll} + & (\tau_R^{*})^{\sigma}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}\frac{\eta^{*}(z_{min}^{*})^{\eta^{*}}}{\eta^{*}-\sigma+1}\left(\Xi_{Z^{D*}}\right)^{-\eta^{*}+\sigma-1}\Xi_{A^{F*}} \\ + & (\tau_R^{*})^{\sigma}\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{\tau^{*}}\right)^{\sigma-1}\frac{\eta^{*}(z_{min}^{*})^{\eta^{*}}}{\eta^{*}-\sigma+1}\left[\left(\Xi_{Z^{X*}}\right)^{-(\eta^{*}-\sigma+1)}-\left(\Xi_{Z^{I*}}\right)^{-(\eta^{*}-\sigma+1)}\right]\Xi_{A^{F}} \\ + & (\tau_R)^{\sigma}\left(\frac{1}{\sigma}\right)\left(\frac{\rho}{(1-s_V)}\right)^{\sigma-1}\frac{\eta^{*}(z_{min}^{*})^{\eta^{*}}}{\eta^{*}-\sigma+1}\left(\Xi_{Z^{I*}}\right)^{-\eta^{*}+\sigma-1}\Xi_{A^{F}} \\ - & f^{D*}\left(z_{min}^{*}\right)^{\eta^{*}}\left(\Xi_{Z^{D*}}\right)^{-\eta^{*}} \\ - & f^{X*}\left(z_{min}^{*}\right)^{\eta^{*}}\left[\left(\Xi_{Z^{X*}}\right)^{-\eta^{*}}-\left(\Xi_{Z^{I*}}\right)^{-\eta^{*}}\right] \\ - & f^{I}\left(z_{min}^{*}\right)^{\eta^{*}}\left(\Xi_{Z^{I*}}\right)^{-\eta^{*}} \end{array} \right)$$

3 Tax on Labor Income and Home Welfare

This section replicates the proposition 3 in Chor (2009) by using the notations in the main text.

3.1 Home Welfare

$$\begin{aligned} V &= L(1-\tau_L) + \left(\tfrac{1}{\theta}-1\right)\left(C^H\right)^\theta + \left(\tfrac{1}{\theta}-1\right)\left(C^F\right)^\theta \\ &= \left(\tfrac{1}{\theta}-1\right)\left(C^F\right)^\theta - L\tau_L + \left[L + \left(\tfrac{1}{\theta}-1\right)\left(C^H\right)^\theta\right] \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &\left(\tfrac{1}{\theta}-1\right)\left(C^F\right)^\theta - L\tau_L \\ &= \left(\tfrac{1}{\theta}-1\right)\left(\Xi_{C^F}\right)^{\frac{1}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}} M^* - \left(\tfrac{s_V(1-s_V)^{-\sigma}(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)\left(\Xi_{Z^{D*}}\right)^{\eta^*+\frac{\kappa(\sigma-1)\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{I*}}\right)^{-\eta^*+\sigma-1}\left(\Xi_{C^F}\right)^{\frac{\kappa(\sigma-1)}{1-\kappa\eta^*}} M^* \\ &= \left(\tfrac{1}{\theta}-1\right)\left(\Xi_{C^F}\right)^{\frac{1}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}} M^* - \left(\tfrac{s_V(1-s_V)^{-\sigma}(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}} M^* \\ &= \left[\left(\tfrac{1}{\theta}-1\right) - \left(\tfrac{s_V(1-s_V)^{-\sigma}(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\left(\Xi_{C^F}\right)^{-1}\right]\left(\Xi_{C^F}\right)^{\frac{1}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}} M^* \end{aligned}$$

We want to prove

- $V \to -\infty$ when $s_V \to 1^-$,
- $\frac{\partial V}{\partial s_V} > 0$ when $s_V = 0$,
- $\frac{\partial V}{\partial s_V} > 0$ when $1-\tau^* < s_V < 0$, that is, $1 < 1-s_V < \tau^*$.

3.2 Prove $V \to -\infty$ when $s_V \to 1^-$.

$$\begin{aligned} V &= \left(\tfrac{1-\theta}{\theta}\right)\left(C^F\right)^\theta - L\tau_L + \left[L + \left(\tfrac{1}{\theta}-1\right)\left(C^H\right)^\theta\right] \\ &= \left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}\left[\left(\tfrac{1}{\theta}-1\right) - \left(\tfrac{s_V(\rho)^\sigma\eta^*}{(1-s_V)^\sigma(\eta^*-\sigma+1)}\right)\left(\tfrac{1}{\Psi_{Z^{I*}}}\right)^{\frac{\eta^*-\sigma+1}{\sigma-1}}\left(\tfrac{1}{\Xi_{C^F}}\right)\right]\left(\Xi_{C^F}\right)^{\frac{1}{1-\kappa\eta^*}} M^* + \left[L + \left(\tfrac{1}{\theta}-1\right)\left(C^H\right)^\theta\right] \end{aligned}$$

where $C^H = \Xi_{C^H}(M)^{\frac{1}{\theta}}$ is not dependent on s_V. Note $(1-s_V)^{-(\sigma-1)} = \left(\frac{1}{1-s_V}\right)^{\sigma-1} \to \infty$. In what follows, we will show the convergence/divergence of the following terms:

- In $(M^*)^{(-\kappa)\eta^*}$, we have the term, $\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{\Xi_{C^F}}\right)^{\frac{(-\kappa)\eta^*}{1-\kappa\eta^*}} \to 0$. This implies that M^* approaches a finite number since $s_V \to 1^-$ implies $\Xi_{C^F} \to \infty$.

- In V, we have the term $\left(\frac{1}{(1-s_V)^{\sigma}}\right)\left(\frac{1}{\Psi_{Z^{I*}}}\right)^{\frac{\eta^*-\sigma+1}{\sigma-1}}\left(\frac{1}{\Xi_{C^F}}\right) \to \infty$

Since $s_V \to 1^-$ implies $\Xi_{C^F} \to \infty$ and M^* converges to a finite number, we establish $V \to -\infty$. Here we have defined

$$\begin{aligned}
\Xi_{Z^{D*}} &= (\rho)^{-1}(\sigma)^{\frac{1-\theta}{\theta}}\left(f^{D*}\right)^{\frac{1-\theta}{\theta}}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)^{-\kappa} \\
\Psi_{Z^{X*}} &= \frac{f^{X*}}{(\tau^*)^{-(\sigma-1)}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}} = \left(f^{X*}\right)\sigma(\rho)^{-(\sigma-1)}(\tau^*)^{\sigma-1} \\
\Psi_{Z^{I*}} &= \frac{f^I-f^{X*}}{\left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}} = \left(f^I-f^{X*}\right)\sigma(\rho)^{-(\sigma-1)}\left(\left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{(-1)}\right)
\end{aligned}$$

$$\delta F^{D*}(M^*)^{(-\kappa)\eta^*} = \begin{pmatrix} + & \left(f^{D*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)(z^*_{min})^{\eta^*}\left(\Xi_{Z^{D*}}\right)^{-\eta^*} \\ + & \left(f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)(z^*_{min})^{\eta^*}\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{\Xi_{C^F}}\right)^{\frac{(-\kappa)\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}} \\ + & \left(f^I-f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)(z^*_{min})^{\eta^*}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{\Xi_{C^F}}\right)^{\frac{(-\kappa)\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}} \end{pmatrix}$$

$$\begin{aligned}
\Xi_{C^F} &= \begin{pmatrix} + & (\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & \left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right](\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \end{pmatrix} \\
&= \begin{pmatrix} + & (\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & \left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{\frac{\eta^*}{\sigma-1}}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\left(f^I-f^{X*}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\sigma)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\rho)^{\eta^*}\right) \end{pmatrix}
\end{aligned}$$

3.2.1 The Term $\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{\Xi_{C^F}}\right)^{\frac{(-\kappa)\eta^*}{1-\kappa\eta^*}}$ in $(M^*)^{(-\kappa)\eta^*}$

This term is given by

$$\begin{aligned}
&\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{\Xi_{C^F}}\right)^{\frac{(-\kappa)\eta^*}{1-\kappa\eta^*}} \\
&= \frac{\left(\frac{f^I-f^{X*}}{\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-\eta^*}{\sigma-1}}\left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{\frac{\eta^*}{\sigma-1}}}{\left[(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}+\left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{\frac{\eta^*}{\sigma-1}}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\left(f^I-f^{X*}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\sigma)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\rho)^{\eta^*}\right)\right]^{\frac{(-\kappa)\eta^*}{1-\kappa\eta^*}}} \\
&= \frac{\left(\frac{f^I-f^{X*}}{\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{\eta^*-\sigma+1}{\eta^*}\right)^{\frac{(-\kappa)\eta^*}{1-\kappa\eta^*}}}{\left[\left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{\frac{-(1-\kappa\eta^*)}{(\sigma-1)(-\kappa)}}\left(\frac{\rho}{\tau^*}\right)^{\sigma-1}\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}+\left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{\frac{1-\kappa\eta^*}{(-\kappa)\eta^*}}\left[\sigma\left(f^I-f^{X*}\right)\right]^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\rho)^{\eta^*}\right]^{\frac{(-\kappa)\eta^*}{1-\kappa\eta^*}}}
\end{aligned}$$

Since $(1-s_V)^{-(\sigma-1)} = \left(\frac{1}{1-s_V}\right)^{\sigma-1} \to \infty$, the term, $(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}} \left(\frac{1}{\Xi_{C^F}}\right)^{\frac{(-\kappa)\eta^*}{1-\kappa\eta^*}}$, converges to zero as $s_V \to 1^-$. Note that $s_V \to 1^-$ implies $\Xi_{C^F} \to \infty$. Therefore, we conclude that as $s_V \to 1^-$,

$$\delta F^{D*} (M^*)^{(-\kappa)\eta^*} \to \left(f^{D*}\right) \left(\frac{\sigma-1}{\eta^*-\sigma+1}\right) (z^*_{min})^{\eta^*} (\Xi_{Z^{D*}})^{-\eta^*}$$

3.2.2 The Term $\left(\frac{1}{(1-s_V)^\sigma}\right) \left(\frac{1}{\Psi_{Z^{I*}}}\right)^{\frac{\eta^*-\sigma+1}{\sigma-1}} \left(\frac{1}{\Xi_{C^F}}\right)$

This term is given by

$$\begin{aligned}
&\left(\frac{1}{(1-s_V)^\sigma}\right) \left(\frac{1}{\Psi_{Z^{I*}}}\right)^{\frac{\eta^*-\sigma+1}{\sigma-1}} \left(\frac{1}{\Xi_{C^F}}\right) \\
&= \frac{\left(\frac{1}{(1-s_V)^\sigma}\right)\left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{\frac{\eta^*-\sigma+1}{\sigma-1}} \left(\frac{1}{\sigma}\right)^{\frac{\eta^*-\sigma+1}{\sigma-1}} (\rho)^{\eta^*-\sigma+1} \left(\frac{1}{f^I-f^{X*}}\right)^{\frac{\eta^*-\sigma+1}{\sigma-1}}}{(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} + \left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{\frac{\eta^*}{\sigma-1}} \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left((f^I-f^{X*})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} (\sigma)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} (\rho)^{\eta^*}\right)} \\
&= \frac{\left(\frac{1}{(1-s_V)^\sigma}\right)\left(\frac{1}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}\right)\left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{\frac{\eta^*}{\sigma-1}} \left(\frac{1}{\sigma}\right)^{\frac{\eta^*-\sigma+1}{\sigma-1}} (\rho)^{\eta^*-\sigma+1} \left(\frac{1}{f^I-f^{X*}}\right)^{\frac{\eta^*-\sigma+1}{\sigma-1}}}{(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} + \left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{\frac{\eta^*}{\sigma-1}} \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left((f^I-f^{X*})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} (\sigma)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} (\rho)^{\eta^*}\right)} \\
&= \frac{\left(\frac{\frac{1}{(1-s_V)}}{1-(1-s_V)^{\sigma-1}(\tau^*)^{-\sigma+1}}\right)\left(\frac{1}{\sigma}\right)^{\frac{\eta^*-\sigma+1}{\sigma-1}} (\rho)^{\eta^*-\sigma+1} \left(\frac{1}{f^I-f^{X*}}\right)^{\frac{\eta^*-\sigma+1}{\sigma-1}}}{\frac{(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}}{\left[(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}\right]^{\frac{\eta^*}{\sigma-1}}} + \left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left((f^I-f^{X*})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} (\sigma)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} (\rho)^{\eta^*}\right)}
\end{aligned}$$

Hence, this term goes to ∞ as $s_V \to 1^-$.

3.3 Home Welfare which is differentiated

$$\frac{\partial V}{\partial s_V} = (1-\theta)\left(C^F\right)^{\theta-1}\frac{\partial C^F}{\partial s_V} - \frac{\partial L\tau_L}{\partial s_V} + (1-\theta)\left(C^H\right)^{\theta-1}\frac{\partial C^H}{\partial s_V}$$

where $\frac{\partial C^H}{\partial s_V} = 0$ will be shown below. Therefore we have

$$\frac{\partial\left[\left(\frac{1}{\theta}-1\right)\left(C^F\right)^\theta - L\tau_L\right]}{\partial s_V}$$

$$= \frac{\partial}{\partial s_V}\left(\left[\left(\frac{1}{\theta}-1\right) - \left(\frac{s_V(1-s_V)^{-\sigma}(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\Xi_{C^F})^{-1}\right](\Xi_{C^F})^{\frac{1}{1-\kappa\eta^*}}(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}M^*\right)$$

$$= \left(\begin{array}{l}
-\left[\left(\frac{(1-s_V)^{-\sigma}(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\Xi_{C^F})^{-1}\right](\Xi_{C^F})^{\frac{1}{1-\kappa\eta^*}}(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}M^* \\
-\left[\left(\frac{s_V(1-s_V)^{-\sigma-1}\sigma(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\Xi_{C^F})^{-1}\right](\Xi_{C^F})^{\frac{1}{1-\kappa\eta^*}}(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}M^* \\
-\left[\left(\frac{s_V(1-s_V)^{-\sigma}(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)\frac{-\eta^*+\sigma-1}{\sigma-1}(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}\frac{\partial\Psi_{Z^{I*}}}{\partial s_V}(\Xi_{C^F})^{-1}\right](\Xi_{C^F})^{\frac{1}{1-\kappa\eta^*}}(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}M^* \\
-\left[\left(\frac{s_V(1-s_V)^{-\sigma}(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(-1)(\Xi_{C^F})^{-2}\frac{\partial\Xi_{C^F}}{\partial s_V}\right](\Xi_{C^F})^{\frac{1}{1-\kappa\eta^*}}(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}M^* \\
+\left[\left(\frac{1}{\theta}-1\right)-\left(\frac{s_V(1-s_V)^{-\sigma}(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\Xi_{C^F})^{-1}\right]\frac{1}{1-\kappa\eta^*}(\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\frac{\partial\Xi_{C^F}}{\partial s_V}(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}M^* \\
+\left[\left(\frac{1}{\theta}-1\right)-\left(\frac{s_V(1-s_V)^{-\sigma}(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\Xi_{C^F})^{-1}\right](\Xi_{C^F})^{\frac{1}{1-\kappa\eta^*}}(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}\frac{\partial M^*}{\partial s_V}
\end{array}\right)$$

3.4 Derive all differentiated terms

3.4.1 Note $\frac{\partial\Xi_{C^H}}{\partial s_V} = \frac{\partial\Xi_{Z^D}}{\partial s_V} = \frac{\partial\Xi_{C^{F*}}}{\partial s_V} = \frac{\partial\Xi_{Z^{D*}}}{\partial s_V} = 0$.

3.4.2 Derive $\frac{\partial\Xi_{Z^{I*}}}{\partial s_V}$ and $\frac{\partial\Psi_{Z^{I*}}}{\partial s_V}$.

$$\begin{aligned}
\Xi_{Z^{I*}} &= (\Psi_{Z^{I*}})^{\frac{1}{\sigma-1}}(\Xi_{C^F})^{\frac{-\kappa}{1-\kappa\eta^*}}(\Xi_{Z^{D*}})^{\frac{-\kappa\eta^*}{1-\kappa\eta^*}} \\
\Psi_{Z^{I*}} &= \left(f^I - f^{X*}\right)\sigma(\rho)^{-(\sigma-1)}\left(\left[(1-s_V)^{-(\sigma-1)} - (\tau^*)^{-(\sigma-1)}\right]^{(-1)}\right)
\end{aligned}$$

The first-order differentiation leads to

$$\begin{aligned}
\frac{1}{\Xi_{Z^{I*}}}\frac{\partial\Xi_{Z^{I*}}}{\partial s_V} &= \left[\left(\frac{1}{\sigma-1}\right)\frac{1}{\Psi_{Z^{I*}}}\frac{\partial\Psi_{Z^{I*}}}{\partial s_V} + \left(\frac{-\kappa}{1-\kappa\eta^*}\right)\frac{1}{\Xi_{C^F}}\frac{\partial\Xi_{C^F}}{\partial s_V}\right] \\
\frac{1}{\Psi_{Z^{I*}}}\frac{\partial\Psi_{Z^{I*}}}{\partial s_V} &= \left[(1-s_V)^{-(\sigma-1)} - (\tau^*)^{-(\sigma-1)}\right]^{(-1)}(1-\sigma)(1-s_V)^{-\sigma} \\
&= \frac{-(\sigma-1)(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)} - (\tau^*)^{-(\sigma-1)}}
\end{aligned}$$

3.4.3 Derive $\frac{\partial \Xi_{Z^{X*}}}{\partial s_V}$ and $\frac{\partial \Psi_{Z^{X*}}}{\partial s_V} = 0$

$$\Xi_{Z^{X*}} = \left(\Psi_{Z^{X*}}\right)^{\frac{1}{\sigma-1}} \left(\Xi_{C^F}\right)^{\frac{-\kappa}{1-\kappa\eta^*}} \left(\Xi_{Z^{D*}}\right)^{\frac{-\kappa\eta^*}{1-\kappa\eta^*}}, \qquad \Psi_{Z^{X*}} = \left(f^{X*}\right)\sigma\left(\rho\right)^{-(\sigma-1)}\left(\tau^*\right)^{\sigma-1}$$

The first-order differentiation leads to

$$\frac{1}{\Xi_{Z^{X*}}}\frac{\partial \Xi_{Z^{X*}}}{\partial s_V} = \left(\frac{-\kappa}{1-\kappa\eta^*}\right)\frac{1}{\Xi_{C^F}}\frac{\partial \Xi_{C^F}}{\partial s_V}$$

3.4.4 Derive $\frac{\partial \Xi_{A^F}}{\partial s_V}$

$$\Xi_{A^F} = \left(\Xi_{C^F}\right)^{\frac{\kappa(\sigma-1)}{1-\kappa\eta^*}} \left(\Xi_{Z^{D*}}\right)^{\frac{\kappa(\sigma-1)\eta^*}{1-\kappa\eta^*}}$$

The first-order differentiation leads to

$$\frac{1}{\Xi_{A^F}}\frac{\partial \Xi_{A^F}}{\partial s_V} = \left(\frac{\kappa(\sigma-1)}{1-\kappa\eta^*}\right)\frac{1}{\Xi_{C^F}}\frac{\partial \Xi_{C^F}}{\partial s_V}$$

3.4.5 Derive $\frac{\partial \Xi_{C^F}}{\partial s_V}$

$$\begin{aligned}
\Xi_{C^F} &= \begin{pmatrix} + & \left(\tau^*\right)^{-(\sigma-1)}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ - & \left(\tau^*\right)^{-(\sigma-1)}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & \left(1-s_V\right)^{-(\sigma-1)}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \end{pmatrix} \\
\Psi_{Z^{X*}} &= \left(f^{X*}\right)\sigma\left(\rho\right)^{-(\sigma-1)}\left(\tau^*\right)^{\sigma-1} \\
\Psi_{Z^{I*}} &= \left(f^{I}-f^{X*}\right)\sigma\left(\rho\right)^{-(\sigma-1)}\left(\left[\left(1-s_V\right)^{-(\sigma-1)}-\left(\tau^*\right)^{-(\sigma-1)}\right]^{(-1)}\right)
\end{aligned}$$

Then we can get

$$\begin{aligned}
\frac{\partial \Xi_{C^F}}{\partial s_V} &= \begin{pmatrix} +(\sigma-1)\left(1-s_V\right)^{-\sigma}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}} \\ -\left[\left(1-s_V\right)^{-(\sigma-1)}-\left(\tau^*\right)^{-(\sigma-1)}\right]\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\sigma-1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(\frac{1}{\Psi_{Z^{I*}}}\frac{\partial \Psi_{Z^{I*}}}{\partial s_V}\right) \end{pmatrix} \\
\frac{1}{\Psi_{Z^{I*}}}\frac{\partial \Psi_{Z^{I*}}}{\partial s_V} &= \frac{-(\sigma-1)(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}
\end{aligned}$$

$$\therefore \frac{\partial \Xi_{C^F}}{\partial s_V} = \begin{pmatrix} +(\sigma-1)(1-s_V)^{-\sigma}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}} \\ +(\sigma-1)(1-s_V)^{-\sigma}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\sigma-1}\right)(\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}} \end{pmatrix}$$

$$= (\rho)^{\sigma-1}\left(\frac{(\eta^*)^2}{\eta^*-\sigma+1}\right)(1-s_V)^{-\sigma}(\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}$$

3.4.6 Note $\frac{\partial \Psi_{Z^X}}{\partial s_V} = \frac{\partial \Psi_{Z^I}}{\partial s_V} = 0$

3.4.7 Note $\frac{\partial \Xi_{C^{H*}}}{\partial s_V} = \frac{\partial M}{\partial s_V} = 0$

$$\delta F^D (M)^{-\kappa\eta} = \begin{pmatrix} + & (f^D)\left(\frac{\sigma-1}{\eta-\sigma+1}\right)(z_{min})^{\eta}(\Xi_{Z^D})^{-\eta} \\ + & (f^X)\left(\frac{\sigma-1}{\eta-\sigma+1}\right)(z_{min})^{\eta}(\Psi_{Z^X})^{\frac{-\eta}{\sigma-1}}(\Xi_{C^{H*}})^{\frac{\kappa\eta}{1-\kappa\eta}}(\Xi_{Z^D})^{\frac{\kappa\eta\eta}{1-\kappa\eta}} \\ + & (f^{I*}-f^X)\left(\frac{\sigma-1}{\eta-\sigma+1}\right)(z_{min})^{\eta}(\Psi_{Z^I})^{\frac{-\eta}{\sigma-1}}(\Xi_{C^{H*}})^{\frac{\kappa\eta}{1-\kappa\eta}}(\Xi_{Z^D})^{\frac{\kappa\eta\eta}{1-\kappa\eta}} \end{pmatrix}$$

where $\theta\sigma + 1 - \sigma < 0$ and $\sigma > 1$. We define $\kappa \equiv \frac{\theta\sigma+1-\sigma}{\theta(\sigma-1)} < 0$ and $\theta\sigma + 1 - \sigma = \kappa\theta(\sigma-1)$.

$$\Xi_{Z^D} = (\rho)^{-1}(\sigma)^{\frac{1-\theta}{\theta}}\left(f^D\right)^{\frac{1-\theta}{\theta}}\left(\frac{\eta}{\eta-\sigma+1}\right)^{-\kappa}$$

$$\Xi_{C^{H*}} = \begin{pmatrix} + & (\tau)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta}{\eta-\sigma+1}\right)(\Psi_{Z^X})^{\frac{-\eta+\sigma-1}{\sigma-1}} \\ - & (\tau)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta}{\eta-\sigma+1}\right)(\Psi_{Z^I})^{\frac{-\eta+\sigma-1}{\sigma-1}} \\ + & (\rho)^{\sigma-1}\left(\frac{\eta}{\eta-\sigma+1}\right)(\Psi_{Z^I})^{\frac{-\eta+\sigma-1}{\sigma-1}} \end{pmatrix}$$

$$\Psi_{Z^X} = \left(f^X\right)\sigma(\rho)^{-(\sigma-1)}(\tau)^{\sigma-1}$$

$$\Psi_{Z^I} = \left(f^{I*}-f^X\right)\sigma(\rho)^{-(\sigma-1)}\left(\left[1-(\tau)^{-(\sigma-1)}\right]^{(-1)}\right)$$

where $\theta\sigma + 1 - \sigma < 0$ and $\sigma > 1$. We define $\kappa \equiv \frac{\theta\sigma+1-\sigma}{\theta(\sigma-1)} < 0$ and $\theta\sigma + 1 - \sigma = \kappa\theta(\sigma-1)$. Therefore, there is no dependence on s_V in M and $\Xi_{C^{H*}}$.

3.4.8 Derive $\frac{\partial M^*}{\partial s_V}$

$$\delta F^{D*}\left(M^*\right)^{-\kappa\eta^*}=\left(\begin{array}{ll} + & \left(f^{D*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Xi_{Z^{D*}}\right)^{-\eta^*} \\ + & \left(f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\tau_C\right)^{\frac{-\eta^*\sigma}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}} \\ + & \left(f^I-f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\tau_C\right)^{\frac{-\eta^*\sigma}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}} \end{array}\right)$$

Therefore, we obtain

$$\begin{aligned} & -\delta F^{D*}\kappa\eta^*\left(M^*\right)^{-\kappa\eta^*-1}\frac{\partial M^*}{\partial s_V} \\ = & \left(\begin{array}{ll} + & \left(f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\left(\frac{\kappa\eta^*}{1-\kappa\eta^*}\right)\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}-1}\left(\frac{\partial\Xi_{C^F}}{\partial s_V}\right) \\ + & \left(f^I-f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\frac{-\eta^*}{\sigma-1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{\Psi_{Z^{I*}}}\frac{\partial\Psi_{Z^{I*}}}{\partial s_V}\right)\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}} \\ + & \left(f^I-f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\left(\frac{\kappa\eta^*}{1-\kappa\eta^*}\right)\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}-1}\left(\frac{\partial\Xi_{C^F}}{\partial s_V}\right) \end{array}\right) \end{aligned}$$

where $\theta\sigma+1-\sigma<0$ and $\sigma>1$. We define $\kappa\equiv\frac{\theta\sigma+1-\sigma}{\theta(\sigma-1)}<0$ and $\theta\sigma+1-\sigma=\kappa\theta(\sigma-1)<0$.

$$\begin{aligned} \Xi_{Z^{D*}} &= (\rho)^{-1}(\sigma)^{\frac{1-\theta}{\theta}}\left(f^{D*}\right)^{\frac{1-\theta}{\theta}}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)^{-\kappa} \\ \frac{\partial\Xi_{C^F}}{\partial s_V} &= (\rho)^{\sigma-1}\left(\frac{(\eta^*)^2}{\eta^*-\sigma+1}\right)\left(1-s_V\right)^{-\sigma}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ \frac{1}{\Psi_{Z^{I*}}}\frac{\partial\Psi_{Z^{I*}}}{\partial s_V} &= \frac{-(\sigma-1)(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}} \\ \Psi_{Z^{X*}} &= \left(f^{X*}\right)\sigma(\rho)^{-(\sigma-1)}\left(\tau^*\right)^{\sigma-1} \\ \Psi_{Z^{I*}} &= \left(f^I-f^{X*}\right)\sigma(\rho)^{-(\sigma-1)}\left(\left[\left(1-s_V\right)^{-(\sigma-1)}\left(\tau_R\right)^{\sigma}-\left(\tau^*\right)^{-(\sigma-1)}\right]^{(-1)}\right) \\ \Xi_{C^F} &= \left(\begin{array}{ll} + & \left(\tau^*\right)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left[\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}-\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\right] \\ + & \left(1-s_V\right)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \end{array}\right) \end{aligned}$$

That is,

$$\begin{aligned} & \left(\frac{\eta^*-\sigma+1}{\sigma-1}\right)\left(z^*_{min}\right)^{-\eta^*}\left(\Xi_{C^F}\right)^{1-\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{-\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\delta F^{D*}(-\kappa)\left(M^*\right)^{-\kappa\eta^*-1}\frac{\partial M^*}{\partial s_V} \\ = & \left(\begin{array}{ll} + & \left(f^{X*}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{\kappa}{1-\kappa\eta^*}\right)\left(\frac{\partial\Xi_{C^F}}{\partial s_V}\right) \\ + & \left(f^I-f^{X*}\right)\left(\frac{-1}{\sigma-1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{\Psi_{Z^{I*}}}\frac{\partial\Psi_{Z^{I*}}}{\partial s_V}\right)\Xi_{C^F} \\ + & \left(f^I-f^{X*}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{\kappa}{1-\kappa\eta^*}\right)\left(\frac{\partial\Xi_{C^F}}{\partial s_V}\right) \end{array}\right) \end{aligned}$$

That is,

$$\left(\frac{\eta^*-\sigma+1}{\sigma-1}\right)(z^*_{min})^{-\eta^*}(\Xi_{C^F})(\Xi_{C^F})^{\frac{-\kappa\eta^*}{1-\kappa\eta^*}}(\Xi_{Z^{D*}})^{\frac{-\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\,\delta F^{D*}(-\kappa)(M^*)^{-\kappa\eta^*-1}\,\frac{\partial M^*}{\partial s_V}$$

$$= \left(\begin{array}{cl} + & \left(f^{X*}\right)(\Psi_{Z^{X*}})^{\frac{-\eta^*}{\sigma-1}}\left(\frac{\kappa}{1-\kappa\eta^*}\right)(\rho)^{\sigma-1}\left(\frac{(\eta^*)^2}{\eta^*-\sigma+1}\right)(1-s_V)^{-\sigma}(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & \left(f^I-f^{X*}\right)\left(\frac{-1}{\sigma-1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}\left(\frac{-(\sigma-1)(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}\right)(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{X*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ - & \left(f^I-f^{X*}\right)\left(\frac{-1}{\sigma-1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}\left(\frac{-(\sigma-1)(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}\right)(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & \left(f^I-f^{X*}\right)\left(\frac{-1}{\sigma-1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}\left(\frac{-(\sigma-1)(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}\right)(1-s_V)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & \left(f^I-f^{X*}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}\left(\frac{\kappa}{1-\kappa\eta^*}\right)(\rho)^{\sigma-1}\left(\frac{(\eta^*)^2}{\eta^*-\sigma+1}\right)(1-s_V)^{-\sigma}(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \end{array}\right)$$

$$= \left(\begin{array}{cl} + & \left(f^{X*}\right)(\Psi_{Z^{X*}})^{\frac{-\eta^*}{\sigma-1}}\left(\frac{\kappa\eta^*}{1-\kappa\eta^*}\right)(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(1-s_V)^{-\sigma}(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & \left(f^I-f^{X*}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}\left(\frac{(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}\right)(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{X*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & \left(f^I-f^{X*}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}(1-s_V)^{-\sigma}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & \left(f^I-f^{X*}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}(1-s_V)^{-\sigma}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\left(\frac{\kappa\eta^*}{1-\kappa\eta^*}\right) \end{array}\right)$$

$$= \left(\begin{array}{cl} + & \frac{1}{\sigma}(\rho)^{\sigma-1}(\Psi_{Z^{X*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(1-s_V)^{-\sigma}(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{1}{1-\kappa\eta^*}\right) \\ + & \left(f^I-f^{X*}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{1-\kappa\eta^*}\right)(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(1-s_V)^{-\sigma}(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \end{array}\right)$$

Therefore, we obtain

$$\left(\frac{\eta^*-\sigma+1}{\sigma-1}\right)(z^*_{min})^{-\eta^*}(\Xi_{C^F})(\Xi_{C^F})^{\frac{-\kappa\eta^*}{1-\kappa\eta^*}}(\Xi_{Z^{D*}})^{\frac{-\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\,\delta F^{D*}(-\kappa)(M^*)^{-\kappa\eta^*-1}\,\frac{\partial M^*}{\partial s_V}$$

$$= \left(\begin{array}{cl} + & \frac{1}{\sigma}(\rho)^{\sigma-1}(\Psi_{Z^{X*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(1-s_V)^{-\sigma}(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{1}{1-\kappa\eta^*}\right) \\ + & \left(f^I-f^{X*}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(1-s_V)^{-\sigma}(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\left(\frac{1}{1-\kappa\eta^*}\right) \end{array}\right) > 0$$

Therefore, we conclude

$$\therefore \left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right) = \frac{\Xi_{M_2^*}}{\Xi_{M_1^*}} > 0$$

where we have

$$\begin{aligned}
\Xi_{M_1^*} &\equiv \left(\frac{\eta^*-\sigma+1}{\sigma-1}\right)\left(z^*_{min}\right)^{-\eta^*}\left(\Xi_{C^F}\right)\left(\Xi_{C^F}\right)^{\frac{-\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{-\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\delta F^{D*}(-\kappa)\left(M^*\right)^{-\kappa\eta^*} > 0 \\
\Xi_{M_2^*} &\equiv \left(\begin{array}{l} +\frac{1}{\sigma}\left(\rho\right)^{(\sigma-1)}\left(\Psi_{Z^{X*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(1-s_V\right)^{-\sigma}\left(\tau^*\right)^{-(\sigma-1)}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{1}{1-\kappa\eta^*}\right) \\ +\left(f^I-f^{X*}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(1-s_V\right)^{-\sigma}\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(\frac{1}{1-\kappa\eta^*}\right) \end{array}\right) > 0 \\
&\equiv \left(\begin{array}{l} +\left(f^{X*}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(1-s_V\right)^{-\sigma}\left(\tau^*\right)^{-(\sigma-1)}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{1}{1-\kappa\eta^*}\right) \\ +\left(f^I-f^{X*}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(1-s_V\right)^{-\sigma}\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(\frac{1}{1-\kappa\eta^*}\right) \end{array}\right) > 0
\end{aligned}$$

where we have

$$\begin{aligned}
\Psi_{Z^{X*}} &= \left(f^{X*}\right)\sigma\left(\rho\right)^{-(\sigma-1)}\left(\tau^*\right)^{\sigma-1} \\
\Psi_{Z^{I*}} &= \left(f^I-f^{X*}\right)\sigma\left(\rho\right)^{-(\sigma-1)}\left(\left[\left(1-s_V\right)^{-(\sigma-1)}-\left(\tau^*\right)^{-(\sigma-1)}\right]^{(-1)}\right) \\
\delta F^{D*}\left(M^*\right)^{-\kappa\eta^*} &= \left(\begin{array}{ll} + & \left(f^{D*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Xi_{Z^{D*}}\right)^{-\eta^*} \\ + & \left(f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}} \\ + & \left(f^I-f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}} \end{array}\right) \\
\Xi_{C^F} &= \left(\begin{array}{ll} + & \left(\tau^*\right)^{-(\sigma-1)}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & \left[\left(1-s_V\right)^{-(\sigma-1)}-\left(\tau^*\right)^{-(\sigma-1)}\right]\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \end{array}\right) \\
&= \left(\begin{array}{ll} + & \left(f^{X*}\right)\left(\sigma\right)\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*}{\sigma-1}} \\ + & \left(f^I-f^{X*}\right)\left(\sigma\right)\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}} \end{array}\right)
\end{aligned}$$

Then we can rewrite $\left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right)$ as

$$\begin{aligned}
&\therefore \left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right) \\
&= \left(\begin{array}{l} +\frac{\left(f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\left(\frac{1}{\Xi_{C^F}}\right)\left(1-s_V\right)^{-\sigma}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{1}{1-\kappa\eta^*}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\delta F^{D*}(-\kappa)\left(M^*\right)^{-\kappa\eta^*}} \\ +\frac{\left(f^I-f^{X*}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\left(\frac{1}{\Xi_{C^F}}\right)\left(1-s_V\right)^{-\sigma}\left(\rho\right)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{1}{1-\kappa\eta^*}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\delta F^{D*}(-\kappa)\left(M^*\right)^{-\kappa\eta^*}} \end{array}\right) \\
&= \left(\begin{array}{l} +\frac{\left(f^{X*}\right)\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{\Xi_{C^F}}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\left(1-s_V\right)^{-\sigma}\left(\rho\right)^{\sigma-1}\left(\frac{1}{1-\kappa\eta^*}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\delta F^{D*}(-\kappa)\left(M^*\right)^{-\kappa\eta^*}} \\ +\frac{\left(f^I-f^{X*}\right)\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{\Xi_{C^F}}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)\left(z^*_{min}\right)^{\eta^*}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\left(1-s_V\right)^{-\sigma}\left(\rho\right)^{\sigma-1}\left(\frac{1}{1-\kappa\eta^*}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\delta F^{D*}(-\kappa)\left(M^*\right)^{-\kappa\eta^*}} \end{array}\right)
\end{aligned}$$

Therefore we conclude for $\left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right)$ as

$$\begin{aligned}
&\therefore \left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right)\\
&= \frac{\left(\frac{1}{\sigma}\right)\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)(z^*_{min})^{\eta^*}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}}(1-s_V)^{-\sigma}(\rho)^{\sigma-1}\left(\frac{1}{1-\kappa\eta^*}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}}{\delta F^{D*}(-\kappa)(M^*)^{-\kappa\eta^*}}\\
&= \frac{\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)(z^*_{min})^{\eta^*}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\left(\frac{1}{\sigma}\right)\left(\Psi_{Z^{I*}}\right)(1-s_V)^{-\sigma}(\rho)^{\sigma-1}\left(\frac{1}{1-\kappa\eta^*}\right)}{\delta F^{D*}(-\kappa)(M^*)^{-\kappa\eta^*}}\\
&= \frac{\left[(f^I-f^{X*})\left(\frac{\sigma-1}{\eta^*-\sigma+1}\right)(z^*_{min})^{\eta^*}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}\right)\left(\frac{1}{(-\kappa)(1-\kappa\eta^*)}\right)}{\delta F^{D*}(M^*)^{-\kappa\eta^*}}\\
&< \left(\frac{(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}\right)\left(\frac{1}{(-\kappa)(1-\kappa\eta^*)}\right)
\end{aligned}$$

3.4.9 Note $\frac{1}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}} - \left(\frac{-\kappa}{1-\kappa\eta^*}\right)\left(\frac{1}{(1-s_V)^{-\sigma}}\frac{1}{\Xi_{C^F}}\frac{\partial \Xi_{C^F}}{\partial s_V}\right) > 0$

We can show the positivity of the following term:

$$\begin{aligned}
&\frac{1}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}} - \left(\frac{-\kappa}{1-\kappa\eta^*}\right)\left(\frac{1}{(1-s_V)^{-\sigma}}\frac{1}{\Xi_{C^F}}\frac{\partial \Xi_{C^F}}{\partial s_V}\right)\\
=\;&\frac{1}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}} - \left(\frac{-\kappa}{1-\kappa\eta^*}\right)\left(\frac{1}{\Xi_{C^F}}\right)(\rho)^{\sigma-1}\left(\frac{(\eta^*)^2}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}
\end{aligned}$$

This is equivalent to show the inequality:

$$\Xi_{C^F} > \left((1-s_V)^{-(\sigma-1)} - (\tau^*)^{-(\sigma-1)}\right)\left(\frac{-\kappa}{1-\kappa\eta^*}\right)(\rho)^{\sigma-1}\left(\frac{(\eta^*)^2}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}$$

That is,

$$\begin{aligned}
&(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left[\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} - \left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\right] + (1-s_V)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\\
=\;&(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} + \left((1-s_V)^{-(\sigma-1)} - (\tau^*)^{-(\sigma-1)}\right)(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\\
>\;&\left((1-s_V)^{-(\sigma-1)} - (\tau^*)^{-(\sigma-1)}\right)\left(\frac{-\kappa\eta^*}{1-\kappa\eta^*}\right)(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}
\end{aligned}$$

That is,

$$(\tau^*)^{-(\sigma-1)}\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}} > \left((1-s_V)^{-(\sigma-1)} - (\tau^*)^{-(\sigma-1)}\right)\left(\frac{-1}{1-\kappa\eta^*}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}$$

which is true since the left term is positive and the right term is negative.

3.5 [Cost Side] Derive $\frac{\partial L\tau_L}{\partial s_V}$

From the government budget balance, we have

$$L\tau_L = G_S = M^* (\rho)^\sigma \left(\frac{\eta^*}{\eta^*-\sigma+1}\right) (\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}} s_V (1-s_V)^{-\sigma} (\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}} (\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}$$

where we define $\kappa \equiv \frac{\theta\sigma+1-\sigma}{\theta(\sigma-1)} < 0$ and $\theta\sigma + 1 - \sigma = \kappa\theta(\sigma-1)$. The first-order differentiation leads to

$$\frac{\partial L\tau_L}{\partial s_V} = G_S \left[\frac{1}{s_V} + \frac{\sigma}{(1-s_V)} + \left(\frac{-(\eta^*-\sigma+1)}{\sigma-1}\right) \frac{1}{\Psi_{Z^{I*}}} \frac{\partial \Psi_{Z^{I*}}}{\partial s_V} + \left(\frac{\kappa\eta^*}{1-\kappa\eta^*}\right) \frac{1}{\Xi_{C^F}} \frac{\partial \Xi_{C^F}}{\partial s_V} + \left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right) \right]$$

3.6 [Cost Side] Note $\frac{\partial C^H}{\partial s_V} = 0$

$$\frac{\partial C^H}{\partial s_V} = (\sigma)^{\frac{1}{\theta}} \left(f^D\right)^{\frac{1}{\theta}} \left(\frac{\eta}{\eta-\sigma+1}\right)^{\frac{1}{\theta}} \frac{1}{\theta} (M)^{\frac{1}{\theta}-1} \frac{\partial M}{\partial s_V}$$

where we have

$$C^H = (\sigma)^{\frac{1}{\theta}} \left(f^D\right)^{\frac{1}{\theta}} \left(\frac{\eta}{\eta-\sigma+1}\right)^{\frac{1}{\theta}} (M)^{\frac{1}{\theta}},$$

$$\theta\sigma + 1 - \sigma < 0, \qquad \sigma > 1, \qquad \theta < 1,$$

$$\rho = \frac{\sigma-1}{\sigma}, \qquad \kappa \equiv \frac{\theta\sigma+1-\sigma}{\theta(\sigma-1)} < 0.$$

The mass M does not depend on s_V.

3.7 [Benefit Side] Derive $\frac{\partial C^F}{\partial s_V}$

$$\frac{\partial C^F}{\partial s_V} = \frac{C^F}{\theta(1-\kappa\eta^*)\Xi_{C^F}} \left(\frac{\partial \Xi_{C^F}}{\partial s_V}\right) + \frac{C^F}{\theta M^*} \left(\frac{\partial M^*}{\partial s_V}\right)$$

where we have

$$C^F = (\Xi_{C^F})^{\frac{1}{\theta-\theta\kappa\eta^*}} (\Xi_{Z^{D*}})^{\frac{\eta^*}{\theta-\theta\kappa\eta^*}} (M^*)^{\frac{1}{\theta}}$$

$$\theta\sigma + 1 - \sigma < 0, \qquad \sigma > 1, \qquad \theta < 1, \qquad \rho = \frac{\sigma-1}{\sigma}, \qquad \kappa \equiv \frac{\theta\sigma+1-\sigma}{\theta(\sigma-1)} < 0.$$

3.8 Derive $\frac{\partial V}{\partial s_V}$

We have obtained

$$
\begin{aligned}
-\frac{\partial L\tau_L}{\partial s_V} &= -G_S\left[\ \frac{1}{s_V}+\frac{\sigma}{(1-s_V)}+\left(\frac{(-\eta^*+\sigma-1)}{\sigma-1}\right)\frac{1}{\Psi_{ZI*}}\frac{\partial \Psi_{ZI*}}{\partial s_V}+\left(\frac{\kappa\eta^*}{1-\kappa\eta^*}\right)\frac{1}{\Xi_{C^F}}\frac{\partial \Xi_{C^F}}{\partial s_V}+\left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right)\ \right]\\
\frac{\partial C^F}{\partial s_V} &= \frac{C^F}{\theta(1-\kappa\eta^*)\Xi_{C^F}}\left(\frac{\partial \Xi_{C^F}}{\partial s_V}\right)+\frac{C^F}{\theta M^*}\left(\frac{\partial M^*}{\partial s_V}\right)\\
\frac{\partial C^H}{\partial s_V} &= 0
\end{aligned}
$$

Hence, the indirect utility is given by

$$
\begin{aligned}
&\frac{\partial V}{\partial s_V} = -\frac{\partial L\tau_L}{\partial s_V}+(1-\theta)\left(C^F\right)^{\theta-1}\frac{\partial C^F}{\partial s_V}\\
&= \left(\begin{array}{l}
-G_S\left[\ \frac{1}{s_V}+\frac{\sigma}{(1-s_V)}+\left(\frac{(-\eta^*+\sigma-1)}{\sigma-1}\right)\frac{1}{\Psi_{ZI*}}\frac{\partial \Psi_{ZI*}}{\partial s_V}+\left(\frac{\kappa\eta^*}{1-\kappa\eta^*}\right)\frac{1}{\Xi_{C^F}}\frac{\partial \Xi_{C^F}}{\partial s_V}+\left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right)\ \right]\\
+(1-\theta)\left(C^F\right)^{\theta}\frac{1}{\theta(1-\kappa\eta^*)\Xi_{C^F}}\left(\frac{\partial \Xi_{C^F}}{\partial s_V}\right)+(1-\theta)\left(C^F\right)^{\theta}\frac{1}{\theta M^*}\left(\frac{\partial M^*}{\partial s_V}\right)
\end{array}\right)\\
&= \left(\begin{array}{l}
-\left[\left(\frac{(\rho)^{\sigma}\eta^*}{\eta^*-\sigma+1}\right)\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}s_V\left(1-s_V\right)^{-\sigma}\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}M^*\right]\frac{1}{s_V}\\
-\left[\left(\frac{(\rho)^{\sigma}\eta^*}{\eta^*-\sigma+1}\right)\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}s_V\left(1-s_V\right)^{-\sigma}\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}M^*\right]\frac{\sigma}{(1-s_V)}\\
-\left[\left(\frac{(\rho)^{\sigma}\eta^*}{\eta^*-\sigma+1}\right)\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}s_V\left(1-s_V\right)^{-\sigma}\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}M^*\right]\left(\frac{(-\eta^*+\sigma-1)}{\sigma-1}\right)\frac{1}{\Psi_{ZI*}}\frac{\partial \Psi_{ZI*}}{\partial s_V}\\
-\left[\left(\frac{(\rho)^{\sigma}\eta^*}{\eta^*-\sigma+1}\right)\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}s_V\left(1-s_V\right)^{-\sigma}\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}M^*\right]\left(\frac{\kappa\eta^*}{1-\kappa\eta^*}\right)\frac{1}{\Xi_{C^F}}\frac{\partial \Xi_{C^F}}{\partial s_V}\\
-\left[\left(\frac{(\rho)^{\sigma}\eta^*}{\eta^*-\sigma+1}\right)\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}s_V\left(1-s_V\right)^{-\sigma}\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}M^*\right]\left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right)\\
+(1-\theta)\left(C^F\right)^{\theta}\frac{1}{\theta(1-\kappa\eta^*)\Xi_{C^F}}\left(\frac{\partial \Xi_{C^F}}{\partial s_V}\right)+(1-\theta)\left(C^F\right)^{\theta}\frac{1}{\theta M^*}\left(\frac{\partial M^*}{\partial s_V}\right)
\end{array}\right)
\end{aligned}
$$

where we have

$$
\begin{aligned}
C^F &= \left(\Xi_{C^F}\right)^{\frac{1}{\theta-\theta\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{\theta-\theta\kappa\eta^*}}\left(M^*\right)^{\frac{1}{\theta}}\\
\frac{1}{\Psi_{ZI*}}\frac{\partial \Psi_{ZI*}}{\partial s_V} &= \frac{-(\sigma-1)(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}\\
\frac{\partial \Xi_{C^F}}{\partial s_V} &= (\rho)^{\sigma-1}\left(\frac{(\eta^*)^2}{\eta^*-\sigma+1}\right)\left(1-s_V\right)^{-\sigma}\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}
\end{aligned}
$$

Therefore, we obtain

$$\frac{\partial V}{\partial s_V} =$$

$$\left(\begin{array}{l}
-\left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}(\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)M^*(\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}(1-s_V)^{-\sigma} \\
-\left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}(\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)M^*(\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}(1-s_V)^{-\sigma}\,\frac{\sigma s_V}{1-s_V} \\
-\left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}(\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)M^*(\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}(1-s_V)^{-\sigma}\,\frac{(\eta^*-\sigma+1)(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}(s_V) \\
-\left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}(\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)M^*(\Psi_{Z^{I*}})^{\frac{-2(\eta^*-\sigma+1)}{\sigma-1}}\left(\frac{\kappa\eta^*}{1-\kappa\eta^*}\right)\frac{1}{\Xi_{C^F}}(\rho)^{\sigma-1}\left(\frac{(\eta^*)^2}{\eta^*-\sigma+1}\right)s_V(1-s_V)^{-2\sigma} \\
-\left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}(\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}s_V(1-s_V)^{-\sigma}\left(\frac{\partial M^*}{\partial s_V}\right) \\
+\left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}(\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)M^*(\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}(1-s_V)^{-\sigma}\,\frac{(1-\theta)\eta^*}{\theta(1-\kappa\eta^*)\rho} \\
+\left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}}(\Xi_{C^F})^{\frac{1}{1-\kappa\eta^*}}\right]\frac{(1-\theta)}{\theta}\left(\frac{\partial M^*}{\partial s_V}\right)
\end{array}\right)$$

where we have

$$\Xi_{Z^{D*}} = (\rho)^{-1}(\sigma)^{\frac{1-\theta}{\theta}}\left(f^{D*}\right)^{\frac{1-\theta}{\theta}}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)^{-\kappa}$$

$$\Xi_{C^F} = \left(\begin{array}{ll}
+ & (\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{X*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\
- & (\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\
+ & (1-s_V)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}
\end{array}\right)$$

$$\Psi_{Z^{X*}} = \left(f^{X*}\right)\sigma(\rho)^{-(\sigma-1)}(\tau^*)^{\sigma-1}$$

$$\Psi_{Z^{I*}} = \left(\frac{\left(f^I-f^{X*}\right)\sigma(\rho)^{-(\sigma-1)}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}\right)$$

$$\left(\frac{\eta^*-\sigma+1}{\sigma-1}\right)(z^*_{min})^{-\eta^*}(\Xi_{C^F})(\Xi_{C^F})^{-\frac{\kappa\eta^*}{1-\kappa\eta^*}}(\Xi_{Z^{D*}})^{\frac{-\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\delta F^{D*}(-\kappa)(M^*)^{-\kappa\eta^*-1}\frac{\partial M^*}{\partial s_V}$$

$$= \left(\begin{array}{ll}
+ & \frac{1}{\sigma}(\rho)^{\sigma-1}(\Psi_{Z^{X*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(1-s_V)^{-\sigma}(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{1}{1-\kappa\eta^*}\right) \\
+ & \left(f^I-f^{X*}\right)(\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{1-\kappa\eta^*}\right)(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(1-s_V)^{-\sigma}(\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}}
\end{array}\right)$$

Under the assumption of $1-s_V<\tau^*$, determine the sign of $\frac{\partial V}{\partial s_V}$ when $s_V=0$ and $s_V<0$.

3.9 Find the sign of $\frac{\partial V}{\partial s_V}$ when $1-\tau^* < s_V < 0$, that is, $1 < 1-s_V < \tau^*$.

$$\frac{\partial V}{\partial s_V} =$$
$$\left(\begin{array}{l}
+\left[\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)M^*\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}(1-s_V)^{-\sigma}(-1)\\
+\left[\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)M^*\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}(1-s_V)^{-\sigma}\frac{\sigma(-s_V)}{1-s_V}\\
+\left[\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)M^*\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}(1-s_V)^{-\sigma}\frac{(\eta^*-\sigma+1)(1-s_V)^{-\sigma}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}(-s_V)\\
+\left[\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)M^*\left(\Psi_{Z^{I*}}\right)^{\frac{-2(\eta^*-\sigma+1)}{\sigma-1}}\left(\frac{\kappa\eta^*}{1-\kappa\eta^*}\right)\frac{1}{\Xi_{C^F}}(\rho)^{\sigma-1}\left(\frac{(\eta^*)^2}{\eta^*-\sigma+1}\right)(-s_V)(1-s_V)^{-2\sigma}\\
+\left[\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}(-s_V)(1-s_V)^{-\sigma}\left(\frac{\partial M^*}{\partial s_V}\right)\\
+\left[\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}\left(\Xi_{C^F}\right)^{\frac{\kappa\eta^*}{1-\kappa\eta^*}}\right]\left(\frac{(\rho)^\sigma\eta^*}{\eta^*-\sigma+1}\right)M^*\left(\Psi_{Z^{I*}}\right)^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}}(1-s_V)^{-\sigma}\frac{(1-\theta)\eta^*}{\theta(1-\kappa\eta^*)\rho}\\
+\left[\left(\Xi_{Z^{D*}}\right)^{\frac{\eta^*}{1-\kappa\eta^*}}\left(\Xi_{C^F}\right)^{\frac{1}{1-\kappa\eta^*}}\right](1-\theta)\frac{1}{\theta}\left(\frac{\partial M^*}{\partial s_V}\right)
\end{array}\right)$$

Therefore, we should only compare the first and sixth terms to determine the sign of $\frac{\partial V}{\partial s_V}$ since all other terms are positive. We obtain $\frac{\partial V}{\partial s_V}|_{(1-\tau^*<s_V<0)} > 0$ since we have $\frac{(1-\theta)\eta^*}{\theta(1-\kappa\eta^*)\rho} - 1 = \frac{\eta^*-\theta(\sigma-1)}{\theta(\sigma-1)-\eta^*(\theta\sigma+1-\sigma)} > 0$ and $(-s_V) > 0$ and

$$\Xi_{Z^{D*}} = (\rho)^{-1}(\sigma)^{\frac{1-\theta}{\theta}}\left(f^{D*}\right)^{\frac{1-\theta}{\theta}}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)^{-\kappa} > 0$$

$$\Xi_{C^F} = \left(\begin{array}{ll}
+ & (\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\\
- & (\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\\
+ & (1-s_V)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}
\end{array}\right) > 0$$

$$\Psi_{Z^{X*}} = \left(f^{X*}\right)\sigma(\rho)^{-(\sigma-1)}(\tau^*)^{\sigma-1} > 0$$

$$\Psi_{Z^{I*}} = \left(\frac{\left(f^I-f^{X*}\right)\sigma(\rho)^{-(\sigma-1)}}{(1-s_V)^{-(\sigma-1)}-(\tau^*)^{-(\sigma-1)}}\right) > 0$$

$$\left(\frac{\eta^*-\sigma+1}{\sigma-1}\right)(z^*_{min})^{-\eta^*}\left(\Xi_{C^F}\right)\left(\Xi_{C^F}\right)^{-\frac{\kappa\eta^*}{1-\kappa\eta^*}}\left(\Xi_{Z^{D*}}\right)^{\frac{-\kappa\eta^*\eta^*}{1-\kappa\eta^*}}\delta F^{D*}(-\kappa)(M^*)^{-\kappa\eta^*-1}\frac{\partial M^*}{\partial s_V}$$
$$= \left(\begin{array}{ll}
+ & \frac{1}{\sigma}(\rho)^{\sigma-1}\left(\Psi_{Z^{X*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}(1-s_V)^{-\sigma}(\tau^*)^{-(\sigma-1)}(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)\left(\frac{1}{1-\kappa\eta^*}\right)\\
+ & \left(f^I-f^{X*}\right)\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*}{\sigma-1}}\left(\frac{1}{1-\kappa\eta^*}\right)(\rho)^{\sigma-1}\left(\frac{\eta^*}{\eta^*-\sigma+1}\right)(1-s_V)^{-\sigma}\left(\Psi_{Z^{I*}}\right)^{\frac{-\eta^*+\sigma-1}{\sigma-1}}
\end{array}\right) > 0$$

and we have

$$\theta\sigma + 1 - \sigma < 0, \qquad \sigma > 1, \qquad \theta < 1, \qquad \rho = \frac{\sigma-1}{\sigma}, \qquad \kappa \equiv \frac{\theta\sigma+1-\sigma}{\theta(\sigma-1)} < 0.$$

3.10 Derive $\frac{\partial V}{\partial s_V}|_{s_V=0}$ by feeding $s_V = 0$ into all derivatives and terms.

$$\frac{\partial V}{\partial s_V}|_{s_V=0}$$

$$= \left(\begin{array}{l} - \left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}} (\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}} \right] \left(\frac{(\rho)^\sigma \eta^*}{\eta^*-\sigma+1} \right) M^* (\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}} \\ + \left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}} (\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}} \right] (1-\theta) (M^*) (\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}} \frac{1}{\theta(1-\kappa\eta^*)} (\rho)^{\sigma-1} \left(\frac{(\eta^*)^2}{\eta^*-\sigma+1} \right) \\ + \left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}} (\Xi_{C^F})^{\frac{1}{1-\kappa\eta^*}} \right] (1-\theta) \frac{1}{\theta} \left(\frac{\partial M^*}{\partial s_V} \right) \end{array} \right)$$

$$= \left(\begin{array}{l} + \left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}} (\Xi_{C^F})^{\frac{\kappa\eta^*}{1-\kappa\eta^*}} \right] \left(\frac{(\rho)^\sigma \eta^*}{\eta^*-\sigma+1} \right) (M^*) (\Psi_{Z^{I*}})^{\frac{-(\eta^*-\sigma+1)}{\sigma-1}} \left[\frac{(1-\theta)\eta^*}{\theta(1-\kappa\eta^*)\rho} - 1 \right] \\ + \left[(\Xi_{Z^{D*}})^{\frac{\eta^*}{1-\kappa\eta^*}} (\Xi_{C^F})^{\frac{1}{1-\kappa\eta^*}} \right] \frac{(1-\theta)}{\theta} \left(\frac{\partial M^*}{\partial s_V} \right) \end{array} \right)$$

where we have

$$\Xi_{Z^{D*}} = (\rho)^{-1} (\sigma)^{\frac{1-\theta}{\theta}} \left(f^{D*} \right)^{\frac{1-\theta}{\theta}} \left(\frac{\eta^*}{\eta^*-\sigma+1} \right)^{-\kappa}$$

$$\Xi_{C^F} = \left(\begin{array}{ll} + & (\tau^*)^{-(\sigma-1)} (\rho)^{\sigma-1} \left(\frac{\eta^*}{\eta^*-\sigma+1} \right) (\Psi_{Z^{X*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ - & (\tau^*)^{-(\sigma-1)} (\rho)^{\sigma-1} \left(\frac{\eta^*}{\eta^*-\sigma+1} \right) (\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \\ + & (\rho)^{\sigma-1} \left(\frac{\eta^*}{\eta^*-\sigma+1} \right) (\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \end{array} \right)$$

$$\Psi_{Z^{X*}} = \left(f^{X*} \right) \sigma (\rho)^{-(\sigma-1)} (\tau^*)^{\sigma-1}$$

$$\Psi_{Z^{I*}} = \left(\frac{\left(f^I - f^{X*} \right) \sigma (\rho)^{-(\sigma-1)}}{1-(\tau^*)^{-(\sigma-1)}} \right)$$

$$\left(\frac{\eta^*-\sigma+1}{\sigma-1} \right) (z^*_{min})^{-\eta^*} (\Xi_{C^F}) (\Xi_{C^F})^{-\frac{\kappa\eta^*}{1-\kappa\eta^*}} (\Xi_{Z^{D*}})^{\frac{-\kappa\eta^*\eta^*}{1-\kappa\eta^*}} \delta F^{D*}(-\kappa) (M^*)^{-\kappa\eta^*-1} \frac{\partial M^*}{\partial s_V}$$

$$= \left(\begin{array}{ll} + & \frac{1}{\sigma} (\rho)^{\sigma-1} (\Psi_{Z^{X*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} (\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} (\tau^*)^{-(\sigma-1)} (\rho)^{\sigma-1} \left(\frac{\eta^*}{\eta^*-\sigma+1} \right) \left(\frac{1}{1-\kappa\eta^*} \right) \\ + & \left(f^I - f^{X*} \right) (\Psi_{Z^{I*}})^{\frac{-\eta^*}{\sigma-1}} \left(\frac{1}{1-\kappa\eta^*} \right) (\rho)^{\sigma-1} \left(\frac{\eta^*}{\eta^*-\sigma+1} \right) (\Psi_{Z^{I*}})^{\frac{-\eta^*+\sigma-1}{\sigma-1}} \end{array} \right)$$

Therefore, we observe the sign of the following term

$$\frac{(1-\theta)\eta^*}{\theta(1-\kappa\eta^*)\rho} - 1 = \frac{\sigma(1-\theta)\eta^*}{\theta(\sigma-1)-\eta^*(\theta\sigma+1-\sigma)} - 1 = \frac{\sigma(1-\theta)\eta^*-\theta(\sigma-1)+\eta^*(\theta\sigma+1-\sigma)}{\theta(\sigma-1)-\eta^*(\theta\sigma+1-\sigma)} = \frac{\eta^*-\theta(\sigma-1)}{\theta(\sigma-1)-\eta^*(\theta\sigma+1-\sigma)} > 0$$

where parameters are restricted by

$$\sigma > 1 \qquad \tau, \tau^* > 1 \qquad \eta > \sigma - 1 \qquad \eta^* > \sigma - 1 \qquad \theta < 1 \qquad 0 < \rho < 1 \qquad \theta\sigma + 1 - \sigma < 0$$

$$\therefore \eta^* - \theta(\sigma - 1) > \eta^* - \sigma + 1 > 0$$

4 Tax on Labor Income and Foreign Welfare

4.1 Foreign Welfare

$$V^* = L + \left(\tfrac{1}{\theta} - 1\right)\left(C^{F*}\right)^{\theta} + \left(\tfrac{1}{\theta} - 1\right)\left(C^{H*}\right)^{\theta}$$

4.2 Foreign Welfare which is differentiated

$$\frac{\partial V^*}{\partial s_V} = (1-\theta)\left(C^{F*}\right)^{\theta}\left(\frac{1}{C^{F*}}\frac{\partial C^{F*}}{\partial s_V}\right) + (1-\theta)\left(C^{H*}\right)^{\theta}\left(\frac{1}{C^{H*}}\frac{\partial C^{H*}}{\partial s_V}\right)$$

4.3 Derive $\frac{\partial C^{F*}}{\partial s_V}$

$$\begin{aligned}
\frac{1}{C^{F*}}\frac{\partial C^{F*}}{\partial s_V} &= \frac{1}{\theta}\left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right) \\
C^{F*} &= \Xi_{C^{F*}}\left(M^*\right)^{\frac{1}{\theta}} \\
\Xi_{C^{F*}} &= (\sigma)^{\frac{1}{\theta}}\left(f^{D*}\right)^{\frac{1}{\theta}}\left(\frac{\eta^*}{\eta^* - \sigma + 1}\right)^{\frac{1}{\theta}}
\end{aligned}$$

where $\theta\sigma + 1 - \sigma < 0$ and $\sigma > 1$. We define $\kappa \equiv \frac{\theta\sigma+1-\sigma}{\theta(\sigma-1)} < 0$ and $\theta\sigma + 1 - \sigma = \kappa\theta(\sigma - 1)$. We have shown $\frac{\partial M^*}{\partial s_V} > 0$ and therefore $\frac{\partial C^{F*}}{\partial s_V} > 0$.

4.4 Derive $\frac{\partial C^{H*}}{\partial s_V}$

$$
\begin{aligned}
C^{H*} &= \left(\Xi_{C^{H*}}\right)^{\frac{1}{\theta-\theta\kappa\eta}} \left(\Xi_{Z^D}\right)^{\frac{\eta}{\theta-\theta\kappa\eta}} (M)^{\frac{1}{\theta}} \\
\Xi_{C^{H*}} &= \left(\begin{array}{ll} + & (\tau)^{-(\sigma-1)} (\rho)^{\sigma-1} \left(\frac{\eta}{\eta-\sigma+1}\right) \left(\Psi_{Z^X}\right)^{\frac{-\eta+\sigma-1}{\sigma-1}} \\ - & (\tau)^{-(\sigma-1)} (\rho)^{\sigma-1} \left(\frac{\eta}{\eta-\sigma+1}\right) \left(\Psi_{Z^I}\right)^{\frac{-\eta+\sigma-1}{\sigma-1}} \\ + & (\rho)^{\sigma-1} \left(\frac{\eta}{\eta-\sigma+1}\right) \left(\Psi_{Z^I}\right)^{\frac{-\eta+\sigma-1}{\sigma-1}} \end{array} \right) \\
\Psi_{Z^X} &= \frac{f^X}{(\tau)^{-(\sigma-1)}\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}} = \left(f^X\right) \sigma \, (\rho)^{-(\sigma-1)} (\tau)^{\sigma-1} \\
\Psi_{Z^I} &= \frac{f^{I*}-f^X}{\left[1-(\tau)^{-(\sigma-1)}\right]\left(\frac{1}{\sigma}\right)(\rho)^{\sigma-1}} = \left(f^{I*}-f^X\right) \sigma \, (\rho)^{-(\sigma-1)} \left(\left[1-(\tau)^{-(\sigma-1)}\right]^{(-1)} \right) \\
\Xi_{Z^D} &= (\rho)^{-1} (\sigma)^{\frac{1-\theta}{\theta}} \left(f^D\right)^{\frac{1-\theta}{\theta}} \left(\frac{\eta}{\eta-\sigma+1}\right)^{-\kappa}
\end{aligned}
$$

where $\theta\sigma + 1 - \sigma < 0$ and $\sigma > 1$. We define $\kappa \equiv \frac{\theta\sigma+1-\sigma}{\theta(\sigma-1)} < 0$ and $\theta\sigma + 1 - \sigma = \kappa\theta(\sigma - 1)$. We have shown $\frac{\partial M}{\partial s_V} = 0$ and $\frac{\partial \Xi_{C^{H*}}}{\partial s_V} = 0$. Therefore, we get $\frac{\partial C^{H*}}{\partial s_V} = 0$.

4.5 Derive $\frac{\partial V^*}{\partial s_V}$

$$
\begin{aligned}
\frac{\partial V^*}{\partial s_V} &= (1-\theta)\left(C^{F*}\right)^{\theta}\left(\frac{1}{C^{F*}}\frac{\partial C^{F*}}{\partial s_V}\right) + (1-\theta)\left(C^{H*}\right)^{\theta}\left(\frac{1}{C^{H*}}\frac{\partial C^{H*}}{\partial s_V}\right) \\
&= \left(\frac{1}{\theta}-1\right)\left(C^{F*}\right)^{\theta}\left(\frac{1}{M^*}\frac{\partial M^*}{\partial s_V}\right) \\
&> 0
\end{aligned}
$$