MAT1010: Calculus for Economic Analysis I

Lecture 2: Limits, Convergence, and Continuity

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Motivation

Motivation: Why do economists need math?

What motivates econ students to be equipped with tools from

- mathematics,
- statistics,
- programming,
- econometrics?

Motivation: Economic Agent

What is a PERSON in structural economic analysis?

- A PERSON is a Constrained, Intertemporal, Stochastic, Optimization Problem.
  - People with purposes, beliefs, constraints.
  - A government with power to tax, spend, redistribute, borrow.
  - Economic agents have technologies for producing goods, services, and capital.
  - Their behaviors are perturbed by stochastic processes describing information flows and economic shocks.
  - An equilibrium describes how diverse purposes and possibilities are reconciled through markets and government regulations.

Source: Thomas Sargent, Computational Challenges in Macroeconomics, "https://youtu.be/VM7UtaR5wHw"

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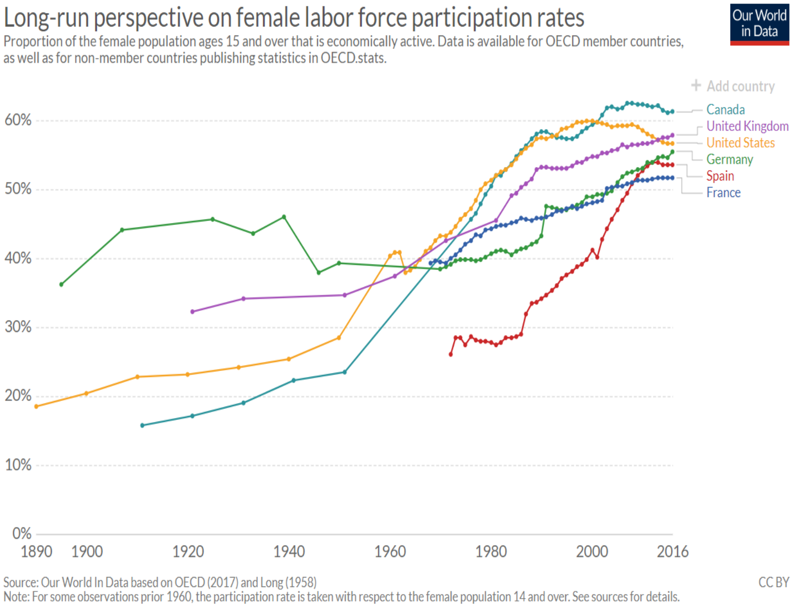
Motivation: Purpose of Structural Economic Analysis

- Interpret HISTORICAL DATA in ways that distinguish CAUSE from COINCIDENCE so that we can ...

- Evaluate Consequences of Alternative Government Policies.

Source: Thomas Sargent, Computational Challenges in Macroeconomics, “https://youtu.be/VM7UtaR5wHw”

Motivation: Female Labor Force Participation



Motivation: Female Labor Force Participation

Consider a 27 year old housewife who has not participated in the job market.

$$\text{Max} \quad UTIL_{27} + (0.96) \times UTIL_{28} + (0.96)^2 \times UTIL_{29} + \cdots + (0.96)^{43} \times UTIL_{70}$$

$$\text{s.t.} \quad UTIL_{27} = CONSUMPTION_{27} + 10000\,(1 - \text{JOIN}_{27})$$

$$CONSUMPTION_{27} = \text{HUSBAND INCOME}_{27} + WAGE_{27} \times \text{JOIN}_{27}$$

$$\log(WAGE_{27}) = \$200 + \$500 \times \text{Education} + \$1000 \times K_{26} - \$25\,(K_{26})^2 + \xi_{27}$$

$$K_{27} = K_{26} + \text{JOIN}_{27} \quad \text{with } K_{26} = 0$$

$$\text{JOIN}_{27} = \begin{cases} 1 & \textit{if} \quad \text{she joins in the labor market and works in a job.} \\ 0 & \textit{if} \quad \text{she does not work.} \end{cases}$$

Motivation: Equity vs. Debt in Corporate Finance

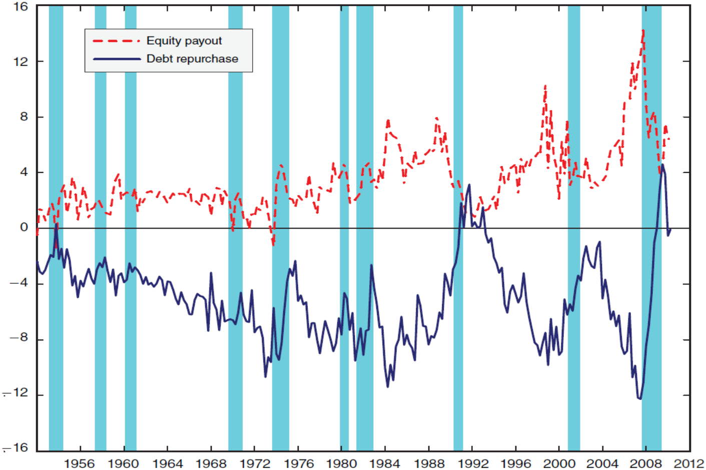


FIGURE 1. FINANCIAL FLOWS IN THE NONFINANCIAL BUSINESS SECTOR (*Corporate and Noncorporate*), 1952:I–2010:II

	Standard deviation(Variable)	Corr(Variable, GDP)
EquPay/GDP	1.13	0.45
DebtRep/GDP	1.46	−0.70

Motivation: Equity vs. Debt in Corporate Finance

What is a firm?

Max $\quad DIV_{2020} + (0.96)DIV_{2021} + (0.96)^2 DIV_{2022} + (0.96)^3 DIV_{2023} + \cdots$

s.t. $\quad DIV_{2020} + I_{2020} + (1.04)BOND_{2019} = \begin{cases} \quad PRICE_{2020} \times \underbrace{(K_{2020})^{\frac{1}{3}} (N_{2020})^{\frac{2}{3}}}_{QUANTITY_{2020}} \\ - \quad WAGE_{2020} \times N_{2020} \\ + \quad BOND_{2020} \end{cases}$

$$K_{2021} = 0.90 \times K_{2020} + I_{2020}$$

$BOND_{2020} < \$5,000,000$ which is susceptible to economic circumstances.

Motivation: Incomes around the world, 2015

- The U.S., Japan, E.U. countries, China, etc have grown quickly, while some nations have stalled.
- A quarter of the world's population lives in countries where the standard of living is lower than it was in the United States in 1900.

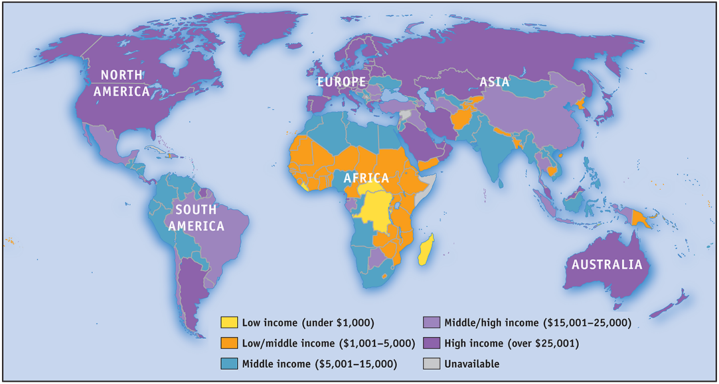


FIGURE 9-2 Krugman/Wells, *Macroeconomics*, 5e, © 2018 Worth Publishers
Data from: World Development Indicators, World Bank.

Motivation: Comparing economies across time and space

- China and India was at the similar level of real GDP per capita from 1950 – 1980.
- What drives China's faster economic growth since 1980?

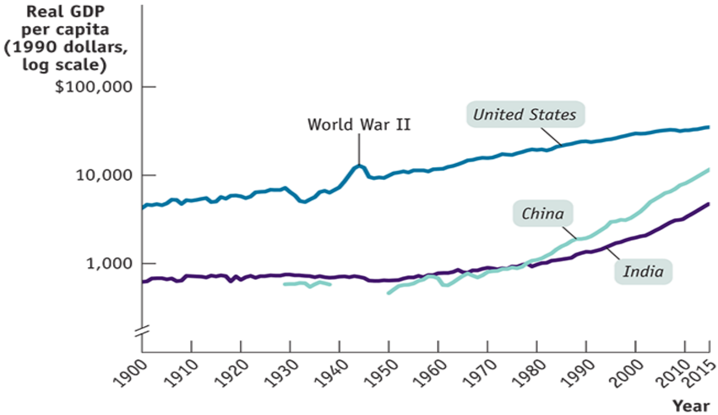
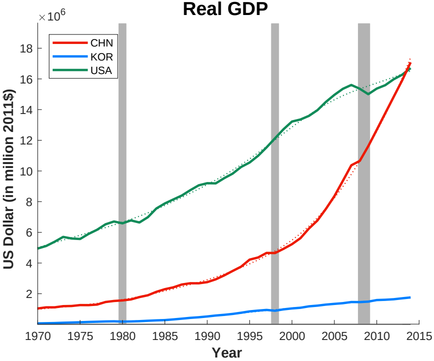


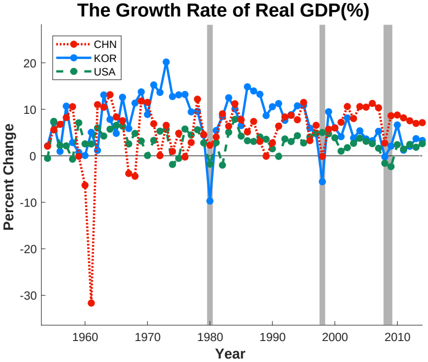
FIGURE 9-1 Krugman/Wells, *Macroeconomics*, 5e, © 2018 Worth Publishers
Data from: Angus Maddison, *Statistics on World Population, GDP, and Per Capita GDP, 1–2008AD*, http://www.ggdc.net/maddison; The Conference Board Total Economy Database™, May 2016, http://www.conference-board.org/data/economydatabase/.

Motivation: Measure an Economy

Closed Economy $C + I + G = Y$

Open Economy $C + I + G + EX - IM = Y$





Source: Penn World Table 9.0

Motivation: The Solow Model: 4 equations and 4 unknowns

- Say, the year is 2000 now. Consider a closed economy and there is no government sector. Workers rent their labor to firms. Their labor is fixed at $L = \overline{L}$.

$$Y_{2000} = C_{2000} + I_{2000}$$

- [The accumulation of capital] Entrepreneurs make investments and produce capital goods through the linear technology:

$$K_{2001} = K_{2000} - \overline{d}K_{2000} + I_{2000} \quad \text{with} \quad \overline{d} = 0.10 = 10\%$$

  - $\overline{d}$ denotes the depreciation rate of capital, which is the ratio of the amount of capital that wears out per year to capital stock.
  - Change in capital stock and investment can be rewritten as:

$$\begin{aligned} \Delta K_{2001} &\equiv K_{2001} - K_{2000} \\ &= I_{2000} - \overline{d}K_{2000} \\ I_{2000} &= \overline{s}Y_{2000} \end{aligned}$$

  where $\overline{s}$ is an exogenous savings rate, say $\overline{s} = 0.30 = 30\%$.

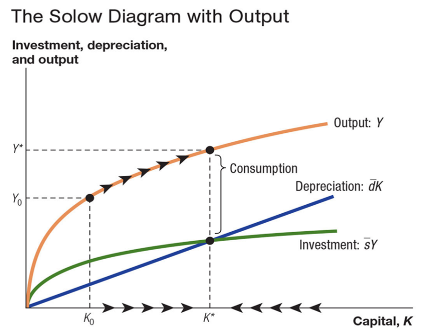
- [Production function] Firms hire capital and labor to produce consumption goods in the perfectly competitive market.

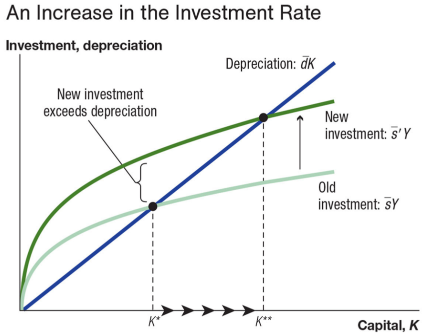
$$Y = F(K,L) = A \cdot K^{\frac{1}{3}} L^{\frac{2}{3}} \qquad i.e. \qquad \frac{Y}{L} = A\left(\frac{K}{L}\right)^{\alpha} \quad \text{where} \quad \alpha = \frac{1}{3}$$

That is, we solve for $Y_{2000}, C_{2000}, I_{2000}, K_{2001}$ with $L_{2000} = \overline{L}$ and K_{2000} given.

Motivation: The Solow Model

- There is no modern problem in social science to be solvable by paper and pencil.
- **Learn how to use Excel.**
- Using advanced programming skills is highly appreciated: Matlab, Python, Stata, C/C++, Fortran, etc.





Limits of Sequences

$[SHS]$ Chap 7.11 $[HWT]$ Chap 9.1 & 9.2

Sequence

- A **sequence** is an enumerated collection of objects.
- Unlike a set, the same elements can appear multiple times at different positions in a sequence since order matters.

$$\{a_n\}_{n=1}^{4} = \{a_1, a_2, a_3, a_4\} = \{A, N, N, E\}$$

$$\{b_i\}_{i=1}^{4} = \{b_1, b_2, b_3, b_4\} = \{A, R, M, Y\}$$

$$\{c_j\}_{j=1}^{4} = \{j\}_{j=1}^{4} = \{1, 2, 3, 4\}$$

$$\{d_n\}_{n=1}^{5} = \{2n+1\}_{n=1}^{5} =$$

$$\{e_n\}_{n=1}^{6} = \{n^2\}_{n=1}^{6} =$$

$$\{f_n\}_{n=1}^{\infty} = \{log_e(n)\}_{n=1}^{\infty} =$$

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\{c_j\}_{j=1}^{4} &= \{j\}_{j=1}^{4} &&= \{1, 2, 3, 4\} \\
\{d_n\}_{n=1}^{5} &= \{2n+1\}_{n=1}^{5} &&= \{3, 5, 7, 9, 11\} \\
\{e_n\}_{n=1}^{6} &= \{n^2\}_{n=1}^{6} &&= \\
\{f_n\}_{n=1}^{\infty} &= \{log_e(n)\}_{n=1}^{\infty} &&=
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$$\{e_n\}_{n=1}^{6} = \{n^2\}_{n=1}^{6} = \{1, 4, 9, 16, 25, 36\}$$

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$$\{e_n\}_{n=1}^{6} = \{n^2\}_{n=1}^{6} = \{1, 4, 9, 16, 25, 36\}$$

$$\{f_n\}_{n=1}^{\infty} = \{log_e(n)\}_{n=1}^{\infty} = \{ln(1), ln(2), ln(3), ln(4), ln(5), \cdots\}$$

Sequence

- To study single-variable calculus, our focus is on a sequence of real numbers, $x_n \in \mathbb{R}$.
- In general, a sequence, x_n, is defined as a function which maps $z \in \mathbb{Z}$ to $c \in \mathbb{C}$, where z denotes a integer number and $c = a + bi \; \forall a, b \in \mathbb{R}$ denotes a complex number (in time series econometrics, for example).
- A sequence, x_n, is defined to be a function which maps $n \in \mathbb{N}$ to $r \in \mathbb{R}$, where n denotes a natural number and r denotes a real number (in this course).
- Find the first ten elements of the following sequences.

x_n	$= a + bn$	with a = 3 and b = 2	Arithmetic Progression
x_n	$= ab^n$	with a = 1 and b = 2	Geometric Progression
x_{n+2}	$= x_n + x_{n+1}$	with x_1 = 1 and x_2 = 1	Fibonacci Sequence

Limit of a Sequence

$$\lim_{n\to\infty} x_n = c \quad i.e. \quad x_n \to c$$

- As $n \in \mathbb{N}$ diverges to the infinity, the sequence x_n gets close to $c \in \mathbb{R}$ by any measure of closeness.
- c, the limit of a sequence x_n, is the value that the terms of the sequence x_n tend to.
- If such $c \in \mathbb{R}$ exists (i.e. well defined), the sequence x_n is called **convergent**.
- If such $c \in \mathbb{R}$ does not exist, the sequence x_n is said to be **divergent**.
- Find the limit, $\lim_{n\to\infty} x_n$:

$$x_n = c \qquad x_n = \frac{1}{n} \qquad x_n = 10n \qquad x_n = \frac{3n+2}{n^{10}}$$

$$x_n = \frac{n^{10}}{10^n} \qquad x_n = log_e(n) \qquad x_n = \left(1 + \frac{1}{n}\right)^n \qquad x_n = (-1)^n$$

$$x_n = (-1)^n 2^n \qquad x_n = \frac{(-1)^n 2^n}{5^n}$$

Limit of a Sequence

$$\lim_{n\to\infty} x_n = c \iff \forall\epsilon \in \mathbb{R}^+ \left(\exists N \in \mathbb{N} \left(\forall n \in \mathbb{N} \left(n \geq N \Longrightarrow |x_n - c| < \epsilon\right)\right)\right)$$

- For every measure of closeness ϵ, the sequence's terms are eventually placed in an area where the distance between x_n and c is less than ϵ (i.e. ϵ-neighborhood).
- $\forall\epsilon > 0$, there exists a natural number N s.t. for every $n > N$, we have $|x_n - c| < \epsilon$.
- Find such N_1 for $\epsilon = 1$, N_2 for $\epsilon = 0.01$, and N_3 for $\epsilon = 0.0001$:

$$x_n = c = 0.1 \qquad x_n = \frac{1}{n} \qquad x_n = 10n \qquad x_n = \frac{3n+2}{n^{10}}$$

$$x_n = \frac{n^{10}}{10^n} \qquad x_n = log_e(n) \qquad x_n = \left(1 + \frac{1}{n}\right)^n \qquad x_n = (-1)^n$$

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- $\forall \epsilon > 0$, there exists a natural number N s.t. for every $n > N$, we have $|x_n - c| < \epsilon$.
- Prove

$$\lim_{n\to\infty} \frac{1}{2^n} = 0, \quad \lim_{n\to\infty} \frac{1}{n} = 0, \quad \neg\left(\lim_{n\to\infty} (-1)^n = 1\right), \quad \lim_{n\to\infty} \sqrt[n]{n} = 1, \quad \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$$

- Hint for (3):

$$\neg\left(\forall \epsilon \in \mathbb{R}^+ \left(\exists N \in \mathbb{N} \left(\forall n \in \mathbb{N} \left(n \geq N \implies |x_n - c| < \epsilon\right)\right)\right)\right)$$

$$\Rightarrow \quad \exists \epsilon \in \mathbb{R}^+ \left(\forall N \in \mathbb{N} \left(\exists n \in \mathbb{N} \left(n \geq N \implies |x_n - c| \geq \epsilon\right)\right)\right)$$

- Hint for (4): $h_n \equiv \sqrt[n]{n} - 1 \geq 0$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} n &= (1+h_n)^n = {}_nC_0 + {}_nC_1 h_n + {}_nC_2 h_n^2 + {}_nC_3 h_n^3 + \cdots + {}_nC_{n-1} h_n^{n-1} + {}_nC_n h_n^n \\ &\geq {}_nC_2 h_n^2 = \frac{n!}{2!(n-2)!} h_n^2 \text{ for all } n \in \mathbb{N} \text{ where } 0! = 1 \end{aligned}$$

Limit of a Sequence

$$\lim_{n\to\infty} x_n = c \iff \forall\epsilon \in \mathbb{R}^+ \left(\exists N \in \mathbb{N} \left(\forall n \in \mathbb{N} \left(n \geq N \implies |x_n - c| < \epsilon\right)\right)\right)$$

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Limit of a Sequence

Theorem

A sequence x_n is bounded if $\exists M \in \mathbb{R}$ such that $|x_n| \leq M \quad \forall n \in \mathbb{N}$.

If a sequence x_n is convergent, then x_n is bounded.

Proof.

For any $\epsilon \in \mathbb{R}^+$, $\exists N$ s.t. $n > N \Rightarrow |x_n - c| < \epsilon$ by assumption. Suppose we find such N^* for $\epsilon = 1$. For $n > N^*$, $|x_n| - |c| \leq |x_n - c| < 1$ because for all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$. Thus, $|x_n| < 1 + |c|$ for all $n > N^*$. Set M to:

$$M = \max \{|x_1|, |x_2|, |x_3|, \cdots, |x_{N^*-2}|, |x_{N^*-1}|, |x_{N^*}|, 1 + |c|\}.$$

Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. □

- The converse does not hold. Observe $x_n = (-1)^n$.

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Limit of a Sequence

Lemma

$$\text{If } \lim_{n\to\infty} x_n = c, \text{then } \forall a > c, \exists N \in \mathbb{N} \text{ such that } n > N \implies x_n < a$$

$$\text{If } \lim_{n\to\infty} x_n = c, \text{then } \forall b < c, \exists N \in \mathbb{N} \text{ such that } n > N \implies x_n > b$$

Theorem

The limit of a convergent sequence is unique.

That is, if $\lim_{n\to\infty} x_n = c$ and $\lim_{n\to\infty} x_n = d$, then $c = d$.

- Prove all above!
- (Hint for the proof of the theorem) Suppose $\lim_{n\to\infty} x_n = c$ and $\lim_{n\to\infty} x_n = d$ with $c < d$. Let $a \equiv \frac{c+d}{2}$. Show $\exists N$ such that $\forall n > N \implies x_n < a$ and $x_n > a$, which is a contradiction.

Limit of a Sequence

Limit Theorem

Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ with $\alpha \in \mathbb{R}$.

$$\lim_{n\to\infty} (x_n \pm y_n) = x \pm y$$

$$\lim_{n\to\infty} (\alpha x_n) = \alpha x$$

$$\lim_{n\to\infty} (x_n y_n) = xy$$

$$\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \frac{x}{y} \text{ for } y_n, y \neq 0$$

- Find $\lim_{n\to\infty} \frac{n-1}{n}$ and $\lim_{n\to\infty} \frac{2n^2+1}{n(n-1)+3}$.
- Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Compare $\lim_{n\to\infty}(x_n + y_n)$ with $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$.

Limit of a Sequence

Theorem

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \text{ and } x_n \geq y_n \text{ for } \forall n \in \mathbb{N} \text{ implies } x \geq y.$$

$$a \leq x_n \leq b \text{ for } \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} x_n = x \text{ implies } a \leq x \leq b.$$

Sandwich Lemma

Let $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$.

$$\lim_{n\to\infty} x_n = c \text{ and } \lim_{n\to\infty} z_n = c \text{ implies } \lim_{n\to\infty} y_n = c.$$

- Prove the theorem by contradiction: suppose $x < y$ and suppose $x < a$ or $x > b$.
- The limit does not preserve strict inequality. See limits of $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$.
- Use the Sandwich Lemma to prove

$$\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n!} = 0, \lim_{n\to\infty} \frac{1}{2^n} = 0 \left(show \ \frac{1}{2^n} < \frac{1}{n} \right), \quad \lim_{n\to\infty} \frac{e^n}{n!} = 0, \lim_{n\to\infty} \frac{n!}{n^n} = 0, \quad \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} = 1$$

Limit of a Sequence

Sandwich Lemma

Let $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$.

$$\lim_{n\to\infty} x_n = c \text{ and } \lim_{n\to\infty} z_n = c \quad \text{implies} \quad \lim_{n\to\infty} y_n = c.$$

Proof.

For any positive real number $\epsilon \in \mathbb{R}^+$, we can always pick up some $\xi, \delta \in \mathbb{Q}$ such that $0 < \xi, \delta < \epsilon$ since the rationals, $\mathbb{Q}$, are a densely ordered subset of the real numbers, $\mathbb{R}$. Then $\exists N_x \in \mathbb{N}$ and $\exists N_z \in \mathbb{N}$ such that

$$n \geq N_x \implies |x_n - c| < \xi \quad \text{and} \quad n \geq N_z \implies |z_n - c| < \delta.$$

Let's define $N \equiv \max(N_x, N_z)$. Therefore, for $\forall n \geq N = \max(N_x, N_z)$,

$$c - \epsilon < c - \xi < x_n < c + \xi < c + \epsilon,$$
$$c - \epsilon < c - \delta < z_n < c + \delta < c + \epsilon,$$

$$\therefore \quad c - \epsilon < x_n \leq y_n \leq z_n < c + \epsilon \implies |y_n - c| < \epsilon.$$

□

Limit of a Sequence

Sandwich Lemma

Let $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$.

$$\lim_{n\to\infty} x_n = c \text{ and } \lim_{n\to\infty} z_n = c \quad \text{implies} \quad \lim_{n\to\infty} y_n = c.$$

Proof.

For any positive real number $\epsilon \in \mathbb{R}^+$, we can always pick up some $\xi, \delta \in \mathbb{Q}$ such that $0 < \xi, \delta < \epsilon$ since the rationals, $\mathbb{Q}$, are a densely ordered subset of the real numbers, $\mathbb{R}$. Then $\exists N_x \in \mathbb{N}$ and $\exists N_z \in \mathbb{N}$ such that

$$n \geq N_x \implies |x_n - c| < \xi \quad \text{and} \quad n \geq N_z \implies |z_n - c| < \delta.$$

Let's define $N \equiv \max(N_x, N_z)$. Therefore, for $\forall n \geq N = \max(N_x, N_z)$,

$$\begin{aligned} c - \epsilon < c - \xi < x_n < c + \xi < c + \epsilon, \\ c - \epsilon < c - \delta < z_n < c + \delta < c + \epsilon, \end{aligned}$$

$$\therefore \quad c - \epsilon < x_n \leq y_n \leq z_n < c + \epsilon \implies |y_n - c| < \epsilon.$$

□

Limit of a Sequence

Definition: Monotone Sequence

x_n is increasing	if $x_1 \leq x_2 \leq x_3 \leq \cdots$.
x_n is strictly increasing	if $x_1 < x_2 < x_3 < \cdots$.
x_n is decreasing	if $x_1 \geq x_2 \geq x_3 \geq \cdots$.
x_n is strictly decreasing	if $x_1 > x_2 > x_3 > \cdots$.
x_n is (strictly) monotone	if it is either (strictly) increasing or (strictly) decreasing.

Definition: Boundedness

A sequence x_n is bounded	if $\exists M \in \mathbb{R}$	such that $\lvert x_n \rvert \leq M$	$\forall n \in \mathbb{N}$.
A sequence x_n is bounded above	if $\exists M \in \mathbb{R}$	such that $x_n \leq M$	$\forall n \in \mathbb{N}$.
A sequence x_n is bounded below	if $\exists M \in \mathbb{R}$	such that $x_n \geq M$	$\forall n \in \mathbb{N}$.

Limit of a Sequence

Properties

If x_n is bounded above (below), then $-x_n$ is bounded below (above).

If x_n is increasing (decreasing), then $-x_n$ is decreasing (increasing).

If x_n is increasing (decreasing), then x_n is bounded below (above) (by x_1).

If M is an upper bound of x_n, then for any $M' \geq M$, M' is also an upper bound.

If M is an lower bound of x_n, then for any $M' \leq M$, M' is also an lower bound.

- Determine monotonicity, and find upper and lower bounds:

$$x_n = \frac{1}{n}, \qquad x_n = n(-1)^n, \qquad x_n = \frac{8^n}{n!}, \qquad x_n = \frac{n!}{n^n}$$

Limit of a Sequence

Definition

$$\lim_{n\to\infty} x_n = \infty \quad \Longleftrightarrow \quad \text{For any real number M} > 0, \exists N \text{ s.t. } \forall n > N \Longrightarrow x_n > M$$

$$\lim_{n\to\infty} x_n = -\infty \quad \Longleftrightarrow \quad \text{For any real number M} < 0, \exists N \text{ s.t. } \forall n > N \Longrightarrow x_n < M$$

If $\lim_{n\to\infty} x_n = \pm\infty$, then x_n is said to be unbounded.

- Determine the convergence of the harmonic sequence. Find $\lim_{n\to\infty} x_n$.

$$x_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

- If x_n is increasing (decreasing) and unbounded, then $x_n \to \infty(-\infty)$.

Limit of a Sequence

[MCT] Monotone Convergence Theorem for Sequences

Every increasing sequence bounded above always converges.

Every decreasing sequence bounded below always converges.

Therefore, every bounded monotone sequence always converges.

- By MCT, if a sequence is monotone and bounded, then it converges.
- Suppose $x_{n+1} = \sqrt{2 + x_n}$ with $x_1 = 0, x_2 = \sqrt{2}$. Prove x_n converges by using MCT. Find its limit.
- Determine the convergence of $x_n = \left(1 + \frac{1}{n}\right)^n$ by using MCT. Note that $x_n \geq 0$ and use $n! \geq 2^{n-1}$ and

$$x_n = {}_nC_0 + {}_nC_1 \tfrac{1}{n} + {}_nC_2 \tfrac{1}{n^2} + {}_nC_3 \tfrac{1}{n^3} + \cdots + {}_nC_{n-1} \tfrac{1}{n^{n-1}} + {}_nC_n \tfrac{1}{n^n}.$$

- Note that $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$.

Limit of a Sequence

[MCT] Monotone Convergence Theorem for Sequences

Every increasing sequence bounded above always converges.

Every decreasing sequence bounded below always converges.

Proof.

Suppose x_n is increasing and bounded above. Then there exists an M such that $x_n \leq M$ for all n. Let S be the least upper bound such that for any $\epsilon > 0$, $S - \epsilon$ is not an upper bound of x_n. Thus, there exists an N such that $x_N > S - \epsilon$. Therefore, for all $n > N$, we have $S - \epsilon < x_N \leq x_n \leq S$. Therefore, $|x_n - S| < \epsilon$ for all $n > N$, that is, $\lim_{n\to\infty} x_n = S$. The MCT for decreasing sequences can be proven analogously. □

- Every increasing sequence bounded above converges to its least upper bound (supremum).
- Every decreasing sequence bounded below converges to its greatest lower bound (infimum).

The Continuous Function Theorem for Sequences

Let $\{x_n\}$ be a sequence of real numbers. If $x_n \to c$ and if f is a function which is continuous at c and defined at all x_n, then $f(x_n) \to f(c)$.

Limit of a Sequence

[MCT] Monotone Convergence Theorem for Sequences

Every increasing sequence bounded above always converges.

Every decreasing sequence bounded below always converges.

Proof.

Suppose x_n is increasing and bounded above. Then there exists an M such that $x_n \leq M$ for all n. Let S be the least upper bound such that for any $\epsilon > 0$, $S - \epsilon$ is not an upper bound of x_n. Thus, there exists an N such that $x_N > S - \epsilon$. Therefore, for all $n > N$, we have $S - \epsilon < x_N \leq x_n \leq S$. Therefore, $|x_n - S| < \epsilon$ for all $n > N$, that is, $\lim_{n\to\infty} x_n = S$. The MCT for decreasing sequences can be proven analogously. □

- Every increasing sequence bounded above converges to its least upper bound (supremum).
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The Continuous Function Theorem for Sequences

Let $\{x_n\}$ be a sequence of real numbers. If $x_n \to c$ and if f is a function which is continuous at c and defined at all x_n, then $f(x_n) \to f(c)$.

Limit of a Sequence

How do we prove any sequence is convergent when we don't know its limit?

Definition: Cauchy Sequence

x_n is a cauchy sequence if $\exists N \in \mathbb{N}$ s.t. $|x_n - x_m| < \epsilon$ for $\forall n, m > N$ with $\forall \epsilon \in \mathbb{R}^+$.

Theorem

(a) Every converging sequence is a cauchy sequence.

(b) Every cauchy sequence is convergent.

Proof.

(a) Suppose $x_n \to x$ as $n \to \infty$. For any $\frac{\epsilon}{2} > 0$ given, there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \frac{\epsilon}{2}$ for $n > N$ and $|x_m - x| < \frac{\epsilon}{2}$ for $m > N$. Then, we have

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The proof of (b) requires many steps: (1) a cauchy seq. is bounded; (2) a bounded seq. has a converging subseq.; (3) when a subseq. of a cauchy seq. converges, then the cauchy seq. converges. Let's skip. □

- Show $x_n = \sum_{k=1}^{n} \frac{1}{k}$ is not a cauchy sequence.

Limit of a Sequence

How do we prove any sequence is convergent when we don't know its limit?

Definition: Cauchy Sequence

x_n is a cauchy sequence if $\exists N \in \mathbb{N}$ s.t. $|x_n - x_m| < \epsilon$ for $\forall n, m > N$ with $\forall \epsilon \in \mathbb{R}^+$.

Theorem

(a) Every converging sequence is a cauchy sequence.

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Proof.

(a) Suppose $x_n \to x$ as $n \to \infty$. For any $\frac{\epsilon}{2} > 0$ given, there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \frac{\epsilon}{2}$ for $n > N$ and $|x_m - x| < \frac{\epsilon}{2}$ for $m > N$. Then, we have

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The proof of (b) requires many steps: (1) a cauchy seq. is bounded; (2) a bounded seq. has a converging subseq.; (3) when a subseq. of a cauchy seq. converges, then the cauchy seq. converges. Let's skip. □

- Show $x_n = \sum_{k=1}^{n} \frac{1}{k}$ is not a cauchy sequence.

Limit of a Sequence: Exercises

- Write out the first ten terms of the sequence:

$$a_1 = 1, a_{n+1} = a_n + (1/2^n) \qquad a_1 = 2, a_{n+1} = (-1)^{n+1} a_n/2 \qquad a_1 = 2, a_2 = -1, a_{n+2} = a_{n+1}/a_n$$

- Find the limit:

$$a_n = \frac{1-2n}{1+2n}, \qquad a_n = \frac{1-5n^4}{n^4+8n^3}, \qquad a_n = \frac{n^2-2n+1}{n-1},$$

$$a_n = \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right), \quad a_n = \left(2-\frac{1}{2^n}\right)\left(3+\frac{1}{1.5^n}\right), \quad a_n = \ln(n) - \ln(n+1)$$

- Find the limit:

$$\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n \qquad \lim_{n\to\infty} x^{\frac{1}{n}} \ \ (x>0) \qquad \lim_{n\to\infty} x^n \ \ (|x|<1)$$

- Find the limit:

$$2, \quad 2+\frac{1}{2}, \quad 2+\frac{1}{2+\frac{1}{2}}, \quad 2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \quad 2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}}, \quad \cdots$$

$$\sqrt{1}, \quad \sqrt{1+\sqrt{1}}, \quad \sqrt{1+\sqrt{1+\sqrt{1}}}, \quad \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}, \quad \cdots$$

$$1, \ 1, \ 1+1, \ 1+2, \ 2+3, \ 3+5, \ 5+8, \ 8+13, \ 13+21, \ 21+34, \ 34+55, \ \cdots$$

See the appendix A.6 of the textbook, **HWT**.

Limits of Functions

$[SHS]$ Chap 6.5 & 7.9 $[HWT]$ Chap 2.2 & 2.3 & 2.4

Limit of a Function

$$\lim_{x \to x_0} f(x) = L$$

- Suppose $f(x)$ is defined on an open interval about x_0, except possibly at x_0 itself.
- If $f(x)$ is arbitrarily close to L for all x sufficiently close to x_0, then we say, the function $f(x)$ approaches **the limit** L as x approaches x_0.
- That is, the limit of $f(x)$ as x approaches x_0 is L.
- [Question] Consider a function $f(x) = x + 1$ and a function $g(x) = \frac{x^2-1}{x-1}$. Are these two functions the same?

Limit of a Function

$$\lim_{x \to x_0} f(x) = L$$

- "The limit of $f(x)$ as x approaches x_0 is L" is formally stated as follows:
  1. Suppose $f(x) : \mathbb{R} \to \mathbb{R}$ is defined on an open interval (a, b) in $\mathbb{R}$ and $x_0 \in (a, b)$ is a point of that interval.
  2. The function $f(x)$ converges to L as x approaches x_0 if
  3. for every real $\epsilon > 0$, there exists a real $\delta > 0$ such that for every x on the interval $0 < |x - x_0| < \delta$ with $x_0 \in (a, b)$, it satisfies $|f(x) - L| < \epsilon$.
- $\lim_{x \to x_0} f(x) = L$, the limit of a function at x_0, is a local property, which is defined on $(x_0 - \delta, x_0 + \delta)$ for some small $\delta \in \mathbb{R}^+$.
  - Prove $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Theorem

If $f(x)$ converges as $x \to x_0$,

there exists a $\delta > 0$ such that $f(x)$ is bounded on $(x_0 - \delta, x_0 + \delta)$.

Theorem

A sequence x_n is bounded if $\exists M \in \mathbb{R}$ such that $|x_n| \leq M \quad \forall n \in \mathbb{N}$.

If a sequence x_n is convergent, then x_n is bounded.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Theorem

If $f(x)$ converges as $x \to x_0$,

there exists a $\delta > 0$ such that $f(x)$ is bounded on $(x_0 - \delta, x_0 + \delta)$.

Theorem

A sequence x_n is bounded if $\exists M \in \mathbb{R}$ such that $|x_n| \leq M \quad \forall n \in \mathbb{N}$.
If a sequence x_n is convergent, then x_n is bounded.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Properties

If $\lim_{x \to x_0} f(x) = L < a$, then $\exists \delta > 0$ such that $\forall x, 0 < |x - x_0| < \delta \Longrightarrow f(x) < a$

If $\lim_{x \to x_0} f(x) = L > b$, then $\exists \delta > 0$ such that $\forall x, 0 < |x - x_0| < \delta \Longrightarrow f(x) > b$

Lemma

If $\lim_{n \to \infty} x_n = c$, then $\forall a > c, \exists N \in \mathbb{N}$ such that $n > N \Longrightarrow x_n < a$

If $\lim_{n \to \infty} x_n = c$, then $\forall b < c, \exists N \in \mathbb{N}$ such that $n > N \Longrightarrow x_n > b$

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Properties

If $\lim_{x \to x_0} f(x) = L < a$, then $\exists \delta > 0$ such that $\forall x, 0 < |x - x_0| < \delta \Longrightarrow f(x) < a$

If $\lim_{x \to x_0} f(x) = L > b$, then $\exists \delta > 0$ such that $\forall x, 0 < |x - x_0| < \delta \Longrightarrow f(x) > b$

Lemma

If $\lim_{n \to \infty} x_n = c$, then $\forall a > c, \exists N \in \mathbb{N}$ such that $n > N \Longrightarrow x_n < a$

If $\lim_{n \to \infty} x_n = c$, then $\forall b < c, \exists N \in \mathbb{N}$ such that $n > N \Longrightarrow x_n > b$

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Limit Theorem for Functions

Suppose $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B$ with $\alpha, \beta \in \mathbb{R}$.

$$\lim_{x\to x_0} (\alpha f(x) + \beta g(x)) = \alpha A + \beta B$$

$$\lim_{x\to x_0} f(x)g(x) = AB$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{A}{B} \text{ provided } g(x), B \neq 0 \text{ on } x \in (x_0 - \delta, x_0 + \delta)$$

Limit Theorem

$$\lim_{n\to\infty} (x_n \pm y_n) = \left(\lim_{n\to\infty} x_n\right) \pm \left(\lim_{n\to\infty} y_n\right) \qquad \lim_{n\to\infty} (\alpha x_n) = \alpha \left(\lim_{n\to\infty} x_n\right)$$

$$\lim_{n\to\infty} (x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right) \qquad \lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n} \text{ for } y_n, \lim_{n\to\infty} y_n \neq 0$$

Students who are interested in formal proofs can refer to the appendix A.5 of the textbook, **HWT**.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Limit Theorem for Functions

Suppose $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B$ with $\alpha, \beta \in \mathbb{R}$.

$$\lim_{x\to x_0} (\alpha f(x) + \beta g(x)) = \alpha A + \beta B$$

$$\lim_{x\to x_0} f(x)g(x) = AB$$

$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$ provided $g(x), B \neq 0$ on $x \in (x_0 - \delta, x_0 + \delta)$

Limit Theorem

$$\lim_{n\to\infty} (x_n \pm y_n) = \left(\lim_{n\to\infty} x_n\right) \pm \left(\lim_{n\to\infty} y_n\right) \qquad \lim_{n\to\infty} (\alpha x_n) = \alpha \left(\lim_{n\to\infty} x_n\right)$$

$\lim_{n\to\infty} (x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right) \qquad \lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$ for $y_n, \lim_{n\to\infty} y_n \neq 0$

Students who are interested in formal proofs can refer to the appendix A.5 of the textbook, **HWT**.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Theorem

The limit of a function is unique.

That is, if $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} f(x) = B$, then $A = B$.

Theorem

The limit of a convergent sequence is unique.

That is, if $\lim_{n \to \infty} x_n = c$ and $\lim_{n \to \infty} x_n = d$, then $c = d$.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Theorem

The limit of a function is unique.

That is, if $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} f(x) = B$, then $A = B$.

Theorem

The limit of a convergent sequence is unique.

That is, if $\lim_{n \to \infty} x_n = c$ and $\lim_{n \to \infty} x_n = d$, then $c = d$.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Theorem

If $f(x)$ converges as $x \to x_0$, then for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\forall\, x_1$ and x_2, $0 < |x_1 - x_0| < \delta$ and $0 < |x_2 - x_0| < \delta$ imply $|f(x_1) - f(x_2)| < \epsilon$.

Cauchy Sequence

x_n is convergent $\iff \exists N \in \mathbb{N}$ s.t. $|x_n - x_m| < \epsilon$ for $\forall n, m > N$ with $\forall \epsilon > 0$.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Theorem

If $f(x)$ converges as $x \to x_0$, then for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\forall\ x_1$ and x_2, $0 < |x_1 - x_0| < \delta$ and $0 < |x_2 - x_0| < \delta$ imply $|f(x_1) - f(x_2)| < \epsilon$.

Cauchy Sequence

x_n is convergent $\iff \exists N \in \mathbb{N}$ s.t. $|x_n - x_m| < \epsilon$ for $\forall n, m > N$ with $\forall \epsilon > 0$.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Theorem

Suppose $f(x)$ and $g(x)$ converge as $x \to x_0$.

If there exists a $\delta > 0$ s.t. $f(x) \leq g(x)$ for all x on $(x_0 - \delta, x_0 + \delta)$,

then $\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x)$.

Theorem

$\lim_{n \to \infty} x_n = x$, $\lim_{n \to \infty} y_n = y$, and $x_n \geq y_n$ for $\forall n \in \mathbb{N}$ implies $x \geq y$.

$a \leq x_n \leq b$ for $\forall n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x$ implies $a \leq x \leq b$.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Theorem

Suppose $f(x)$ and $g(x)$ converge as $x \to x_0$.

If there exists a $\delta > 0$ s.t. $f(x) \leq g(x)$ for all x on $(x_0 - \delta, x_0 + \delta)$,

then $\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x)$.

Theorem

$\lim_{n \to \infty} x_n = x$, $\lim_{n \to \infty} y_n = y$, and $x_n \geq y_n$ for $\forall n \in \mathbb{N}$ implies $x \geq y$.

$a \leq x_n \leq b$ for $\forall n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x$ implies $a \leq x \leq b$.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Sandwich Rule

If there exists a $\delta > 0$ s.t.

$f(x) \leq g(x) \leq h(x)$ for all x on $(x_0 - \delta, x_0 + \delta)$ and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$,

then $\lim_{x \to x_0} g(x) = L$.

Sandwich Lemma

Let $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$.

$\lim_{n \to \infty} x_n = c$ and $\lim_{n \to \infty} z_n = c$ implies $\lim_{n \to \infty} y_n = c$.

- Prove $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$ by using the Sandwich Rule.

Limit of a Function: Properties

Use analogies to understand what is new by comparing what you understood.

Sandwich Rule

If there exists a $\delta > 0$ s.t.

$f(x) \leq g(x) \leq h(x)$ for all x on $(x_0 - \delta, x_0 + \delta)$ and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$,

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- Prove $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$ by using the Sandwich Rule.

Limit of a Function: Properties

Definition: Divergence and Unboundedness

If $\exists \epsilon > 0$ such that $\forall \delta > 0$, there exist x_1 and x_2 which satisfy that $0 < |x_1 - x_0| < \delta$ and $0 < |x_2 - x_0| < \delta$ imply $|f(x_1) - f(x_2)| > \epsilon$, then $f(x)$ diverges as $x \to x_0$.

$$\lim_{x \to x_0} f(x) = \infty \quad \textit{iff} \quad \forall M > 0, \exists \delta > 0 \text{ s.t. } \forall x, 0 < |x - x_0| < \delta \Longrightarrow f(x) > M.$$

$$\lim_{x \to x_0} f(x) = -\infty \quad \textit{iff} \quad \forall M < 0, \exists \delta > 0 \text{ s.t. } \forall x, 0 < |x - x_0| < \delta \Longrightarrow f(x) < M.$$

Definition: Unboundedness

$$\lim_{n \to \infty} x_n = \pm\infty \iff \text{For any real number M} > 0, \exists N \text{ s.t. } \forall n > N \Longrightarrow |x_n| > M$$

Limit of a Function: Properties

Definition: Divergence and Unboundedness

If $\exists \epsilon > 0$ such that $\forall \delta > 0$, there exist x_1 and x_2 which satisfy that $0 < |x_1 - x_0| < \delta$ and $0 < |x_2 - x_0| < \delta$ imply $|f(x_1) - f(x_2)| > \epsilon$, then $f(x)$ diverges as $x \to x_0$.

$$\lim_{x \to x_0} f(x) = \infty \quad \textit{iff} \quad \forall M > 0, \exists \delta > 0 \text{ s.t. } \forall x, 0 < |x - x_0| < \delta \Longrightarrow f(x) > M.$$

$$\lim_{x \to x_0} f(x) = -\infty \quad \textit{iff} \quad \forall M < 0, \exists \delta > 0 \text{ s.t. } \forall x, 0 < |x - x_0| < \delta \Longrightarrow f(x) < M.$$

Definition: Unboundedness

$$\lim_{n \to \infty} x_n = \pm\infty \iff \text{For any real number M} > 0, \exists N \text{ s.t. } \forall n > N \Longrightarrow |x_n| > M$$

Limit of a Function: Properties

Definition: Divergence

If $\exists \epsilon > 0$ such that $\forall \delta > 0$, there exist x_1 and x_2 which satisfy that $0 < |x_1 - x_0| < \delta$ and $0 < |x_2 - x_0| < \delta$ imply $|f(x_1) - f(x_2)| > \epsilon$, then $f(x)$ diverges as $x \to x_0$.

- Prove $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ diverges (hint: take $x_1 = \frac{1}{2n\pi + \frac{\pi}{2}}$ and $x_2 = \frac{1}{2n\pi - \frac{\pi}{2}}$).

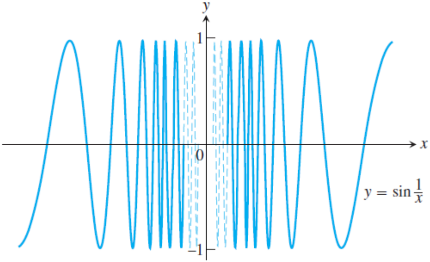


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4).

Limit of a Function: Properties

Definition

$$\lim_{x \to \infty} f(x) = L \;\; \textit{\textbf{iff}} \;\; \forall \epsilon > 0, \exists M > 0 \text{ s.t. } \forall x > M \Longrightarrow |f(x) - L| < \epsilon$$

$$\lim_{x \to -\infty} f(x) = L \;\; \textit{\textbf{iff}} \;\; \forall \epsilon > 0, \exists M < 0 \text{ s.t. } \forall x < M \Longrightarrow |f(x) - L| < \epsilon$$

Limit of a Function: Properties

Definition: Left-hand Limit

$$\lim_{x \to x_0^-} f(x) = L \textit{ iff } \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, x_0 - \delta < x < x_0 \Longrightarrow |f(x) - L| < \epsilon.$$

Definition: Right-hand Limit

$$\lim_{x \to x_0^+} f(x) = L \textit{ iff } \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, x_0 < x < x_0 + \delta \Longrightarrow |f(x) - L| < \epsilon.$$

- True or False: $\lim\limits_{x \to 0} \frac{1}{x} = \infty$ and $\lim\limits_{x \to 0} \frac{1}{x^2} = \infty$?
- Find left-hand and right-hand limits of the sign(signum) function:

$$sgn(x) = \begin{cases} -1 & \textit{if} \quad x < 0 \\ 0 & \textit{if} \quad x = 0 \\ +1 & \textit{if} \quad x > 0 \end{cases}$$

Limit of a Function: Properties

Theorem

$\lim_{x \to x_0} f(x)$ exists if and only if $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exist and they are equal.

[MCT] Monotone Convergence Theorem for Functions

Suppose $f(x)$ is increasing on $[a,b]$. Then, for $\forall x_0 \in (a,b)$, both $\lim_{x \to x_0^-} f(x)$ & $\lim_{x \to x_0^+} f(x)$ exist and $\lim_{x \to x_0^-} f(x) \leq f(x_0) \leq \lim_{x \to x_0^+} f(x)$.

Suppose $f(x)$ is decreasing on $[a,b]$. Then, for $\forall x_0 \in (a,b)$, both $\lim_{x \to x_0^-} f(x)$ & $\lim_{x \to x_0^+} f(x)$ exist and $\lim_{x \to x_0^-} f(x) \geq f(x_0) \geq \lim_{x \to x_0^+} f(x)$.

- Find $\lim_{x \to 0} f(x)$ when $f(x) = \begin{cases} -1 & \textit{if} \quad x < 0 \\ 0 & \textit{if} \quad x = 0 \\ 1 & \textit{if} \quad x > 0 \end{cases}$.

Limit of a Function

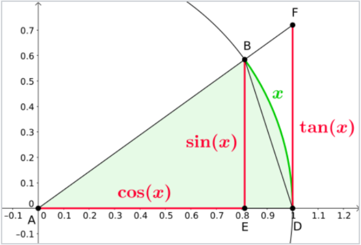
Theorem

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Lemma

$$|\sin(x)| \le |x| \le |\tan(x)| \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Proof.



- Compare areas of triangles with the circle sector, $\frac{x}{2\pi}\pi r^2 = \frac{1}{2}rl$, with $r = 1$ and $l = rx$, and prove the inequalities for $0 < x < \frac{\pi}{2}$.
- Then it follows that $0 < -x < \frac{\pi}{2}$ implies $\sin(-x) < -x < \tan(-x)$.

□

Limit of a Function

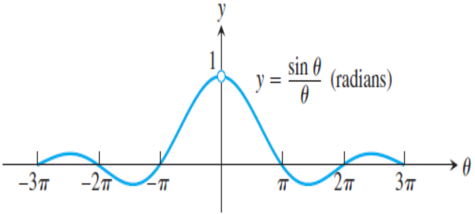
Theorem

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Proof.

For $0 < x < \frac{\pi}{2}$, the inequalities, $\sin(x) < x < \tan(x) = \frac{\sin(x)}{\cos(x)}$, lead to $\cos(x) < \frac{\sin(x)}{x} < 1$. Therefore, $\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1$ by the Sandwich Rule. The left-hand limit, $\lim_{x \to 0^-} \frac{\sin(x)}{x} = 1$ for $-\frac{\pi}{2} < x < 0$, follows by substituting $t \equiv -x$.

□



Limit of a Function

Theorem

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

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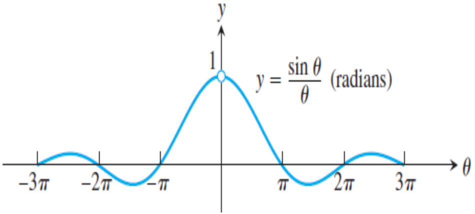
For $0 < x < \frac{\pi}{2}$, the inequalities, $\sin(x) < x < \tan(x) = \frac{\sin(x)}{\cos(x)}$, lead to $\cos(x) < \frac{\sin(x)}{x} < 1$. Therefore, $\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$ by the Sandwich Rule. The left-hand limit, $\lim_{x\to 0^-} \frac{\sin(x)}{x} = 1$ for $-\frac{\pi}{2} < x < 0$, follows by substituting $t \equiv -x$. $\square$



Limit of a Function

Corollary

$$\lim_{x\to x_0} \sin(x) = \sin(x_0) \qquad \lim_{x\to x_0} \cos(x) = \cos(x_0)$$

Proof.

Use trigonometric identity:

$$\sin(x) - \sin(y) = 2\cos\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right), \quad \cos(x) - \cos(y) = -2\sin\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right).$$

Observe that

$$0 \le |\sin(x) - \sin(x_0)| = \left|2\cos\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)\right| \le 2\left|\cos\left(\frac{x+x_0}{2}\right)\right|\left|\sin\left(\frac{x-x_0}{2}\right)\right|$$

$$\le 2\left|\sin\left(\frac{x-x_0}{2}\right)\right| \le 2\left|\frac{x-x_0}{2}\right| = |x-x_0| \text{ for all } x \text{ near } x_0.$$

Therefore, $x \to x_0$ implies $|\sin(x) - \sin(x_0)| \to 0$. We can prove for $\cos(x)$ analogously. □

- Prove $\lim_{x\to x_0} \frac{\sin(x)-\sin(x_0)}{x-x_0} = \cos(x_0)$ and $\lim_{x\to x_0} \frac{\cos(x)-\cos(x_0)}{x-x_0} = -\sin(x_0)$.

Limit of a Function

Corollary

$$\lim_{x \to x_0} \sin(x) = \sin(x_0) \qquad \lim_{x \to x_0} \cos(x) = \cos(x_0)$$

Proof.

Use trigonometric identity:

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$$\leq 2\left|\sin\left(\frac{x-x_0}{2}\right)\right| \leq 2\left|\frac{x-x_0}{2}\right| = |x-x_0| \text{ for all } x \text{ near } x_0.$$

Therefore, $x \to x_0$ implies $|\sin(x) - \sin(x_0)| \to 0$. We can prove for $\cos(x)$ analogously. □

- Prove $\lim_{x \to x_0} \frac{\sin(x) - \sin(x_0)}{x - x_0} = \cos(x_0)$ and $\lim_{x \to x_0} \frac{\cos(x) - \cos(x_0)}{x - x_0} = -\sin(x_0)$.

Limit of a Function

Theorem

For $e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, $\quad e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ with $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proof.

Since $x \to \infty$, restrict x by $x > 1$. Then $\lfloor x \rfloor \le x \le \lfloor x \rfloor + 1$. It follows:

$$g(x) \equiv \left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} \le \left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{x} \le \left(1+\frac{1}{x}\right)^{x} \le \left(1+\frac{1}{\lfloor x\rfloor}\right)^{x} \le \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1} \equiv h(x).$$

Observe $g(x) = \left(1+\frac{1}{n+1}\right)^n$ and $h(x) = \left(1+\frac{1}{n}\right)^{n+1}$ for $x \in [n, n+1)$, where $n \in \mathbb{N}$. Thus,

$$g(x) = \lim_{x\to\infty}\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n} = \lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}}\right] = e$$

$$h(x) = \lim_{x\to\infty}\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)\right] = e$$

□

- Prove $e = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$, $e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$, and $\lim_{x\to x_0}\frac{\ln(x)-\ln(x_0)}{x-x_0} = \frac{1}{x_0}$ (hint: $t \equiv -x$, $\frac{1}{x}$, $\frac{x-x_0}{x_0}$).

Limit of a Function

Theorem

For $e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, $\quad e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ with $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proof.

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$$g(x) \equiv \left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} \le \left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{x} \le \left(1+\frac{1}{x}\right)^{x} \le \left(1+\frac{1}{\lfloor x\rfloor}\right)^{x} \le \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1} \equiv h(x).$$

Observe $g(x) = \left(1+\frac{1}{n+1}\right)^n$ and $h(x) = \left(1+\frac{1}{n}\right)^{n+1}$ for $x \in [n, n+1)$, where $n \in \mathbb{N}$. Thus,

$$g(x) = \lim_{x\to\infty}\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n} = \lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}}\right] = e$$

$$h(x) = \lim_{x\to\infty}\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)\right] = e$$

□

- Prove $e = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$, $e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$, and $\lim_{x\to x_0}\frac{\ln(x)-\ln(x_0)}{x-x_0} = \frac{1}{x_0}$ (hint: $t \equiv -x$, $\frac{1}{x}$, $\frac{x-x_0}{x_0}$).

Limit of a Function: Exercises

- Evaluate:

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5} \qquad \lim_{x\to 1}\frac{x^2+x-2}{x^2-x} \qquad \lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$$

- Determine whether the limits exist:

$$\lim_{x\to 0}\frac{x}{|x|} \qquad \lim_{x\to 1}\frac{1}{x-1}$$

- Prove that $\lim_{x\to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$.

- Find the limits:

$$\lim_{t\to 0}\frac{\sin(kt)}{t} \qquad \lim_{y\to 0}\frac{\sin(3y)}{4y} \qquad \lim_{y\to 0}\frac{\sin(y)}{\sin(2y)}$$

- Let $f(x) = \begin{cases} 3-x, & x < 2 \\ \frac{x}{2}+1, & x > 2 \end{cases}$.

Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2} f(x)$, $\lim_{x\to 4^+} f(x)$, $\lim_{x\to 4^-} f(x)$, and $\lim_{x\to 4} f(x)$.

Continuity of Functions

$[SHS]$ Chap 7.8 & 7.9 & 7.10 $[HWT]$ Chap 2.5

Continuity of a Function

$$\lim_{x \to x_0} f(x) = f(x_0)$$

- Suppose $f(x)$ is defined on an open interval $(x_0 - \delta, x_0 + \delta)$ for some $\delta \in \mathbb{R}^+$.
- A function $f(x)$ is said to be **continuous** at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$ holds.
- Otherwise, we say, $f(x)$ is **discontinuous** at x_0.

▶ Define $\Delta y \equiv f(x) - f(x_0)$ and $\Delta x \equiv x - x_0$.
$f(x)$ is continuous at x_0 if $\Delta x \to 0$ implies $\Delta y \to 0$,
where $\Delta y = f(x) - f(x_0) = f(x_0 + x - x_0) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$.

▶ The limit can get inside of the argument of a function only when the function is continuous at the designated point. That is, $\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right) = f(x_0)$ holds only when $f(x)$ is continuous at x_0 (note that x is continuous on $\mathbb{R}$).

Continuity of a Function

$$\lim_{x \to x_0} f(x) = f(x_0)$$

- **[Continuity Test]** A function $f(x)$ is continuous at an interior point x_0 of its domain if and only if
  1. $f(x_0)$ exists.
  2. $\lim_{x \to x_0} f(x)$ exists, that is, $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$.
  3. $\lim_{x \to x_0} f(x) = f(x_0)$.
- A function $f(x)$ is **right-continuous (continuous from the right)** at x_0 if and only if $\lim_{x \to x_0^+} f(x) = f(x_0)$.
- A function $f(x)$ is **left-continuous (continuous from the left)** at x_0 if and only if $\lim_{x \to x_0^-} f(x) = f(x_0)$.

Continuity of a Function

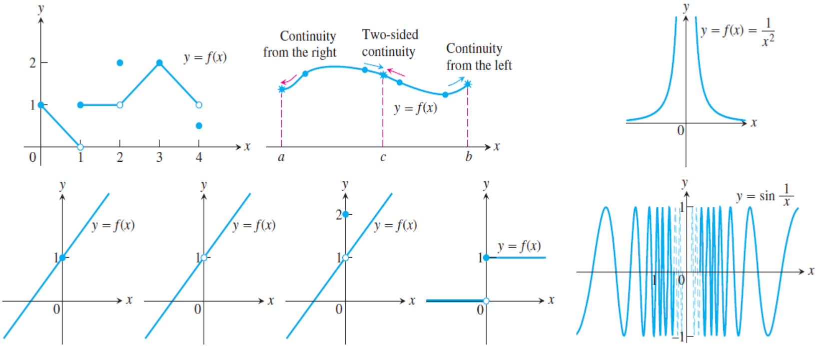
$$\lim_{x \to x_0} f(x) = f(x_0)$$

- A function $f(x)$ is said to be **continuous on an open interval** (a, b) if it is continuous at any $x_0 \in (a, b)$.
- A function $f(x)$ is **continuous on a closed interval** $[a, b]$ if it is continuous on (a, b), right continuous at a, and left continuous at b.

Continuity of a Function: Continuity vs. Discontinuity

- **[Continuity Test]** A function $f(x)$ is continuous at an interior point x_0 of its domain if and only if

  1. $f(x_0)$ exists.
  2. $\lim_{x \to x_0} f(x)$ exists, that is, $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$.
  3. $\lim_{x \to x_0} f(x) = f(x_0)$.



Continuity of a Function: Properties

$$\lim_{x \to x_0} f(x) = f(x_0)$$

- Suppose functions $f(x)$ and $g(x)$ are continuous on an open interval (a,b).
  1. $\alpha f(x) + \beta g(x)$ is continuous on (a,b) with $\alpha, \beta \in \mathbb{R}$.
  2. $f(x)g(x)$ is continuous on (a,b).
  3. $\frac{f(x)}{g(x)}$ is continuous at any $x_0 \in (a,b)$ provided $g(x_0) \neq 0$.
  4. $(f(x))^n$ and $(f(x))^{\frac{1}{n}}$ are continuous at any $x_0 \in (a,b)$ with $n \in \mathbb{N}$.

- Suppose a function $y = f(x)$ is continuous at x_0 and a function $u = g(y)$ is continuous at $y_0 = f(x_0)$. Then $u = g \circ f(x)$ is continuous at x_0.

$$\lim_{x \to x_0} g(f(x)) = g\left(\lim_{x \to x_0} f(x)\right) = g\left(f\left(\lim_{x \to x_0} x\right)\right) = g(f(x_0))$$

- ▶ The composition of two continuous functions is continuous.
- ▶ Continuity is preserved under algebraic operations: $+, -, \times, \div$ (when denominators are nonzero).

Continuity of a Function: Properties

Theorem

Suppose a function $g(y)$ is continuous at the point c and $\lim_{x \to x_0} f(x) = c$. Then we have $\lim_{x \to x_0} g(f(x)) = g(c) = g\left(\lim_{x \to x_0} f(x)\right)$.

Proof.

Let $\epsilon > 0$ be given. Due to the continuity of $g(y)$ at c, there exists a $\xi > 0$ such that

$$\forall y, 0 < |y - c| < \xi \implies |g(y) - g(c)| < \epsilon.$$

Now such ξ is given. Then use the convergence of $f(x)$ towards c as $x \to x_0$ to find some $\delta > 0$ such that

$$\forall x, 0 < |x - x_0| < \delta \implies |f(x) - c| < \xi.$$

Therefore, if we let $y = f(x)$, we have $|g(f(x)) - g(c)| < \epsilon$ whenever $0 < |x - x_0| < \delta$ for all x, which completes the proof. □

Continuity of a Function: Properties

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Suppose a function $g(y)$ is continuous at the point c and $\lim_{x \to x_0} f(x) = c$. Then we have $\lim_{x \to x_0} g(f(x)) = g(c) = g\left(\lim_{x \to x_0} f(x)\right)$.

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Continuity of a Function: Properties

- Any elementary function is continuous in its domain.

Linear functions	$f(x) = mx + b$
Power functions	$f(x) = x^a$
Polynomials	$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
Rational functions	$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 m + b_0}$ with $q(x) \neq 0$
Algebraic functions	Any f(x) from p(x) operated by $+, -, \times, \div, \sqrt{\cdot}, etc$
Cosine/Sine functions	$f(x) = sin(x), cos(x)$
Exponential functions	$f(x) = a^x$ from domain $(-\infty, \infty)$ to range $(0, \infty)$
Logarithmic functions	$f(x) = log_a(x)$ where the base $a \neq 1$ is positive

- How to prove the continuity of x^a, a^x, and $\log_a(x)$ with $a > 0, a \neq 1$?
- x^a, a^x, and $\log_a(x)$ are monotone and surjective, and so continuous.

Continuity of a Function: Properties

- Any elementary function is continuous in its domain.

Linear functions	$f(x) = mx + b$
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Rational functions	$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 m + b_0}$ with $q(x) \neq 0$
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- ► How to prove the continuity of x^a, a^x, and $\log_a(x)$ with $a > 0, a \neq 1$?
- ► x^a, a^x, and $\log_a(x)$ are monotone and surjective, and so continuous.

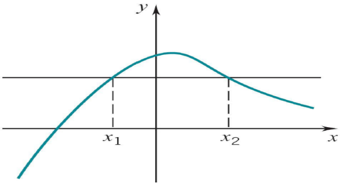
Continuity of a Function: Injection and Surjection

- A function $f : X \to Y$ is **one-to-one (injective)** if

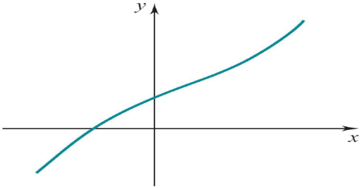
$$x_1 \neq x_2 \quad \text{implies} \quad f(x_1) \neq f(x_2) \quad \forall x_1, x_2 \in X,$$

where the contrapositive is given by

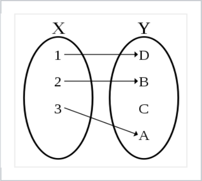
$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2 \quad \forall x_1, x_2 \in X.$$



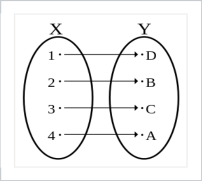
f is not one-to-one: $f(x_1) = f(x_2)$



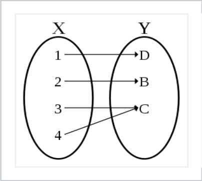
f is one-to-one:



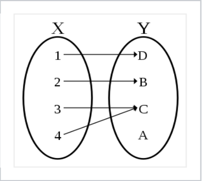
An injective non-surjective function (injection, not a bijection)



An injective surjective function (bijection)



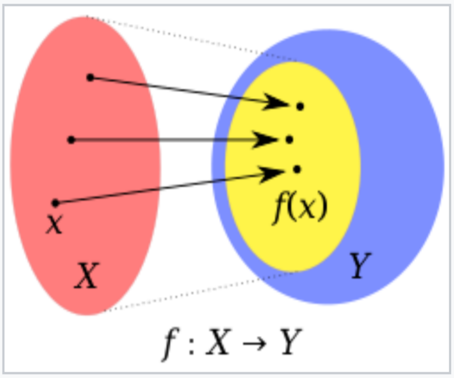
A non-injective surjective function (surjection, not a bijection)



A non-injective non-surjective function (also not a bijection)

Continuity of a Function: Injection and Surjection

- A function f from a set X to a set Y is a **surjection** (an **onto** function) if $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$.
- The function below is not surjective (not an *onto* function) since the mapping (colored yellow) does not fill the whole codomain.



Continuity of a Function: Injection and Surjection

- Is an injective function monotone?

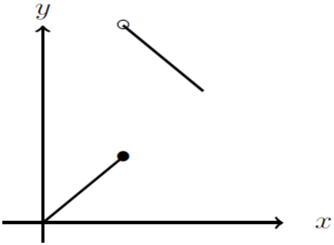
- If a function $f : [a, b]$ is monotone, then is $f : [a, b] \to [min(f(a), f(b)), max(f(a), f(b))]$ surjective?

- If a function $f : [a, b]$ is monotone, then is $f : [a, b] \to [min(f(a), f(b)), max(f(a), f(b))]$ continuous?

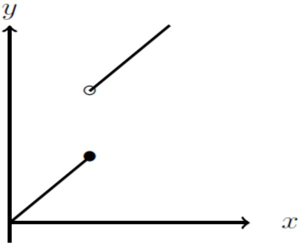
- If a function $f : [a, b]$ is continuous and injective, then is f monotone?

Continuity of a Function: Injection and Surjection

- Is an injective function monotone?
  No, see graph A.
- If a function $f : [a, b]$ is monotone, then is $f : [a, b] \to [min(f(a), f(b)), max(f(a), f(b))]$ surjective?
  No, see graph B.
- If a function $f : [a, b]$ is monotone, then is $f : [a, b] \to [min(f(a), f(b)), max(f(a), f(b))]$ continuous?
  No, see graph B.
- If a function $f : [a, b]$ is continuous and injective, then is f monotone?
  Yes, it is strictly monotone!



Graph A



Graph B

Continuity of a Function: Injection and Surjection

Theorem

If a function $f(x) : [a,b] \to [min(f(a),f(b)), max(f(a),f(b))]$ is a **monotonic surjection**, then $f(x)$ is **continuous** on (a,b).

Proof.

Suppose $f(x)$ is increasing and not continuous at some $x_0 \in (a,b)$. By MCT, the left-hand and right-hand limits exist as $x \to x_0$ and there are three cases for discontinuity at x_0:

(1) $\lim_{x \to x_0^-} f(x) < f(x_0) = \lim_{x \to x_0^+} f(x)$ or

(2) $\lim_{x \to x_0^-} f(x) = f(x_0) < \lim_{x \to x_0^+} f(x)$ or

(3) $\lim_{x \to x_0^-} f(x) < f(x_0) < \lim_{x \to x_0^+} f(x)$. In cases of (1) or (3), there exists some $y' \in \left(\lim_{x \to x_0^-} f(x), f(x_0) \right)$ such that there is no $x' \in (a,b)$ which obtains $y' = f(x')$. Similarly, in cases of (2) or (3), there exists some $y' \in \left(f(x_0), \lim_{x \to x_0^+} f(x) \right)$ such that there is no $x' \in (a,b)$ which assigns $f(x')$ to y'. Both contradict with the definition of surjection. Hence, $f(x)$ is continuous at $x_0 \in (a,b)$. We can analogously prove for $f(x)$ which decreases. □

Continuity of a Function: Injection and Surjection

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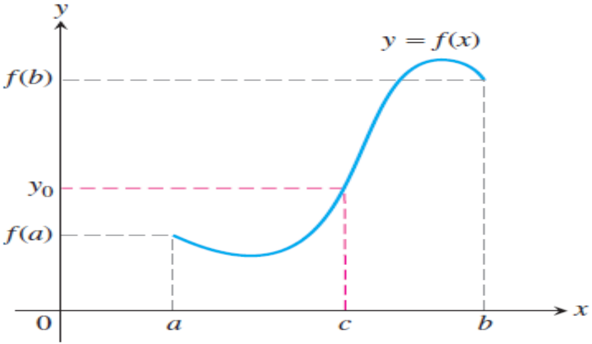
Continuity of a Function: IVT

[IVT] Intermediate Value Theorem

Consider a closed interval $[a, b]$ in $\mathbb{R}$ and a continuous function $f : [a, b] \to \mathbb{R}$.

$\forall y_0 \in [\min(f(a), f(b)), \max(f(a), f(b))], \exists c \in [a, b]$ such that $f(c) = y_0$.

- Prove $f(x) = x^4 - 5x^3 + 3x^2 - 8x - 1 = 0$ has at least two different roots.



Continuity of a Function: Inverse Functions

- **[Question]**

  If a function f is continuous and invertible, then f^{-1} is also continuous?

Definition

Let $f : X \to Y$.

The function f is **invertible** if $\exists! g : Y \to X$ such that $f(x) = y \Leftrightarrow g(y) = x$.

The f inverse, g, is unique and it is denoted by f^{-1}: $Y \to X$.

The f inverse satisfies $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(y) = y$.

- A function f is invertible if and only if it is **bijective**, i.e., both **one-to-one (injective)** and **onto (surjective)**.
- Why does a function f need to be surjective for the existence of its inverse?
- If f inverse is to be a function, any element in its domain should get mapped.
- Non-bijective (either non-injective or non-surjective) functions are not invertible.
- There exists a function which is bijective and discontinuous.
  e.g. $f(x) = x$ if $x \notin \{0, 1\}$, $f(x) = 1$ if $x = 0$, and $f(x) = 0$ if $x = 1$.

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Continuity of a Function: Inverse Functions

Theorem

Suppose a function f is continuous on a closed interval $[a, b]$, where $a < b$.

The f inverse, f^{-1}, exists if and only if f is strictly monotone.

Proof.

$[\Leftarrow]$ A continuous and strictly monotone f is invertible since it is bijective.

$[\Rightarrow]$ $\exists f^{-1}$ implies $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$ $\forall x_1, x_2 \in [a, b]$, and $\therefore f(a) \neq f(b)$, i.e. $f(a) \lessgtr f(b)$.

(1) Suppose f with $f(a) < f(b)$ is not strictly increasing. Then $\exists x_1, x_2 \in (a, b)$ such that $x_1 < x_2$ $\Longrightarrow f(x_1) \geq f(x_2)$.

(1-a) Suppose $f(x_2) \leq f(x_1) < f(b)$. This implies $\exists c \in [x_2, b]$ s.t. $f(x_1) = f(c)$ by IVT where $x_1 < x_2 \leq c$, which is a contradiction to the regularity for the inverse function: $x_1 \neq c \Longrightarrow f(x_1) \neq f(c)$.

(1-b) Suppose $f(a) \; < f(b) \; \leq f(x_1)$. This implies $\exists c' \in [a, x_1]$ s.t. $f(b) = f(c')$ by IVT where $c' \leq x_1 < b$, which is a contradiction to the regularity for the inverse function.

(2) Now suppose f with $f(a) > f(b)$ is not strictly decreasing. We can show the contradiction by analogous steps. □

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Continuity of a Function: Inverse Functions

- **[Question]**

  If a function f is continuous and invertible, then f^{-1} is also continuous?

- What is the answer?
  - If f is continuous and invertible, then f is strictly monotone and so is f^{-1}.
  - Since f^{-1} is strictly monotone and bijective, f^{-1} is also continuous.

Corollary

If a continuous f is strictly monotone, then $\exists f^{-1}$ and so is f^{-1} since it is bijective.

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Continuity of a Function: Exercises

- For what values of a and b is $g(x)$ continuous at every x?

$$g(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

- Prove the following functions are continuous everywhere on their respective domains.

$$y = \sqrt{x^3 - 2x^2 - 5x} \qquad y = \left|\frac{x-2}{x^2-4}\right| \qquad y = \left|\frac{x\sin(x)}{x^2+2}\right|$$

- Show there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

- Use the Intermediate Value Theorem to prove that the equation $\sqrt{2x+5} = 4 - x^2$ has a solution.

Appendix

Limit of a Function: Functions and Sequences

Theorem

If $\lim_{x \to x_0} f(x) = L$,
then, for any sequence x_n with $\lim_{n \to \infty} x_n = x_0$ and $x_n \neq x_0 \ \forall n$, $\lim_{n \to \infty} f(x_n) = L$.

Proof.

For any $\epsilon > 0$, there exists a $\delta > 0$ such that $\forall x, 0 < |x - x_0| < \delta$ implies $|f(x) - L| < \epsilon$.
Taking this δ as given, we can find an N such that $\forall n > N$ implies $|x_n - x_0| < \delta$.
Therefore, $\exists N$ such that $\forall n > N \implies |x_n - x_0| < \delta \implies |f(x_n) - L| < \epsilon$. □

Theorem

Given a sequence x_n, define $f(x) = x_n$ if $x \in [n, n+1)$,

then, $\lim_{n \to \infty} x_n = L$ if and only if $\lim_{x \to \infty} f(x) = L$.

Limit of a Function: Functions and Sequences

Theorem

If $\lim_{x \to x_0} f(x) = L$,
then, for any sequence x_n with $\lim_{n \to \infty} x_n = x_0$ and $x_n \neq x_0 \ \forall n$, $\lim_{n \to \infty} f(x_n) = L$.

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Given a sequence x_n, define $f(x) = x_n$ if $x \in [n, n+1)$,

then, $\lim_{n \to \infty} x_n = L$ if and only if $\lim_{x \to \infty} f(x) = L$.

Logic

- A conditional statement, $P \Longrightarrow Q$: All humans are mammals.
  - If something is a human, it is a mammal.
- Negation, $\neg(P \Longrightarrow Q)$.
  - There exists a human that is not a mammal.
- Inversion (the inverse), $\neg P \Longrightarrow \neg Q$.
  - If something is not a human, it is not a mammal.
- Conversion (the converse), $Q \Longrightarrow P$.
  - If something is a mammal, it is a human.
- Contraposition, $\neg Q \Longrightarrow \neg P$.
  - If something is not a mammal, it is not a human.

Function

- A function f from a set X to a set Y is a rule that assigns a unique element $f(x) \in Y$ to each element $x \in X$.
- The set X of all possible input values is called the **domain** of the function.
- The set Y of all values of $f(x)$ as x varies throughout X is called the **range** of the function.

Linear functions	$f(x) = mx + b$
Power functions	$f(x) = x^a$
Polynomials	$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
Rational functions	$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 m + b_0}$ with $q(x) \neq 0$
Algebraic functions	Any f(x) from p(x) operated by $+, -, \times, \div, \sqrt{\cdot}, etc$
Trigonometric functions	$f(x) = sin(x), cos(x), tan(x) = \frac{sin(x)}{cos(x)}, csc(x), sec(x), cot(x)$
Exponential functions	$f(x) = a^x$ from domain $(-\infty, \infty)$ to range $(0, \infty)$
Logarithmic functions	$f(x) = log_a(x)$ where the base $a \neq 1$ is positive
Transcendental functions	exp, log, (inverse) trigonometric functions, and so on

Function

- Functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions.
- If f and g are functions, then for every $x \in X(f) \cap X(g)$ that belongs to the domains of both f and g,

$$
\begin{aligned}
(f \pm g)(x) &= f(x) \pm g(x) \\
(fg)(x) &= f(x)g(x) \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \text{ at which } g(x) \neq 0 \\
(cf)(x) &= cf(x) \text{ where } x \in X(f)
\end{aligned}
$$

- If f and g are functions, the **composite** function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x))$$

where the domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

Function

- A function $y = f(x)$ is an **even function** of x if $f(x) = f(-x)$ for every x in the function's domain.
- A function $y = f(x)$ is an **odd function** of x if $f(x) = -f(-x)$ for every x in the function's domain.
- A function $f(x)$ is **one-to-one** on a domain X if $x_1 \neq x_2$ in X implies $f(x_1) \neq f(x_2)$.
- A *one-to-one* function intersects each horizontal line at most once.
- For a *one-to-one* function $f(x) : X \to Y$, the inverse function $f^{-1} : Y \to X$ is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b$$

Function

- An absolute value function is given by $f(x) = |x|$.
- A floor function is $f(x) = \lfloor x \rfloor$.
- A ceiling function is $f(x) = \lceil x \rceil$.
- The Dirichlet function is $I_{\mathbb{Q}}(x) = 1$ for $x \in \mathbb{Q}$ and $I_{\mathbb{Q}}(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$, which is nowhere continuous (everywhere discontinuous).

Function

- Does a function whose domain is the empty set exist?
- In set theory terminology, a function $f : X \to Y$ is a subset of $X \times Y$ such that,
  1. For all $x \in X$, there exists a $y \in Y$, such that $(x,y) \in f$.
  2. If $(x,y) \in f$ and $(x,z) \in f$, then $y = z$.
- If X is the empty set, then $X \times Y$ is the empty set. X has no elements and so (1) is true.
- When $X \times Y$ is empty, Any relation from X to Y will have no elements and so (2) is also true.
- Therefore, any relation from the empty set to any other set is a function.

Trigonometric Identities

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}, \quad \cot(x) = \frac{1}{\tan(x)}.$$

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right) \quad \cot(x) = -\tan\left(x + \frac{\pi}{2}\right) \quad \csc(x) = \sec\left(x - \frac{\pi}{2}\right)$$

$$\sin(-x) = -\sin(x) \quad \cos(-x) = \cos(x) \quad \tan(-x) = -\tan(x)$$

$$\sin^2(x) + \cos^2(x) = 1 \quad \sin(2x) = 2\sin(x)\cos(x) \quad \cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} \quad \tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \quad \tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

$$\sin(x)\cos(y) = \frac{\sin(x+y)+\sin(x-y)}{2}$$

$$\cos(x)\cos(y) = \frac{\cos(x+y)+\cos(x-y)}{2} \quad \sin(x)\sin(y) = -\frac{\cos(x+y)-\cos(x-y)}{2}$$

$$\begin{aligned}
\sin(x) + \sin(y) &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\
\sin(x) - \sin(y) &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \\
\cos(x) + \cos(y) &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\
\cos(x) - \cos(y) &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)
\end{aligned}$$

Motivation: Equity vs. Debt in Corporate Finance

[Question] What is the importance of financial shocks – perturbations that originate directly in the financial sector?

- Observe the net payments to equity holders and the net debt repurchases in the nonfinancial business sector (corporate and noncorporate).
- The figure and table documents the cyclical properties of firms' equity and debt flows at an aggregate level. We then build a business cycle model with explicit roles for firms' debt and equity financing that is capable of capturing the empirical cyclical properties of the financial flows.
- Debt is preferred to equity, but the firms' ability to borrow is limited by an enforcement constraint which is subject to random disturbances – financial shocks affecting the firms' ability to borrow.
- Financing comes from three sources: internal funds, debt, and new equities. Companies prioritize their sources of financing, first preferring internal financing, and then debt, lastly raising equity as a "last resort".
- The pecking order theory is popularized by Myers and Majluf (1984) where they argue that equity is a less preferred means to raise capital because when managers (who are assumed to know better about true condition of the firm than investors) issue new equity, investors believe that managers think that the firm is overvalued and managers are taking advantage of this over-valuation. As a result, investors will place a lower value to the new equity issuance.

Motivation: Equity vs. Debt in Corporate Finance

[Question] What is the importance of financial shocks – perturbations that originate directly in the financial sector?

- Equity payout is procyclical and debt payout is countercyclical.
- Financial shocks contributed significantly to the observed dynamics of real and financial variables.
- Equity payout is defined as dividends and share repurchases minus equity issues of nonfinancial corporate businesses, minus net proprietor's investment in noncorporate businesses. This captures the net payments to business owners (shareholders of corporations and noncorporate business owners).
- Debt is defined as "Credit Market Instruments," which include only liabilities that are directly related to credit markets transactions. Debt repurchases are simply the reduction in outstanding debt (or increase if negative). Both variables are expressed as a fraction of business GDP.
- First, equity payouts are negatively correlated with debt repurchases. This suggests that there is some substitutability between equity and debt financing.
- Second, while equity payouts tend to increase in booms, debt repurchases increase during or around recessions. This suggests that recessions lead firms to restructure their financial positions by cutting the growth rate of debt and reducing the payments to shareholders.

Motivation: Measure an Economy

- GDP (Y) is the market value of total production within a nation's border.
- Consumer spending (C) is household spending on goods and services.
- Investment spending (I) is spending on productive physical capital (such as machinery and construction of buildings) and on changes to inventories (total investment equals fixed investment plus the change in inventories).
- Government purchases of goods and services (G) are total expenditures on goods and services by federal, state, and local governments: education, national defense, Social Security, etc.
- Goods and services sold to other countries are exports (EX).
- Goods and services purchased from other countries are imports (IM).

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